Κβαντομηχανική ΙΙ, ΣΕΜΦΕ

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Άσκηση 1.

α)

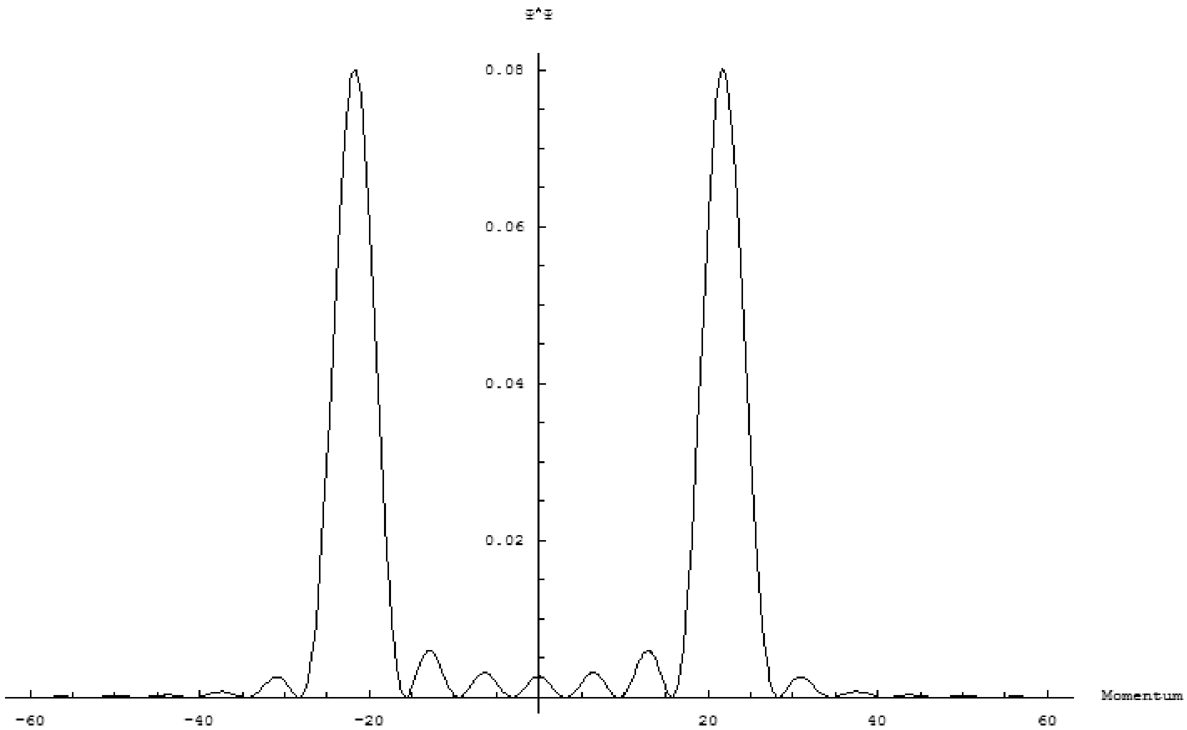
$$
\begin{aligned}
& N^2 \int_0^L \sin^2(\frac{n\pi x}{L})\mathrm{d}x = 1 \\
& \int_0^L \sin^2(\frac{n\pi x}{L})\mathrm{d}x = \int_0^L \frac{1}{2}\mathrm{d}x - \frac{1}{2}\int_0^L \cos(\frac{2n\pi x}{L})\mathrm{d}x = \frac{L}{2} \\
& \Rightarrow N^2\frac{L}{2} = 1 \Rightarrow N = \sqrt{\frac{2}{L}} \qquad \blacksquare
\end{aligned}
$$

β)

$$
\begin{aligned}
& \Psi(x) = \frac{i}{2}\sqrt{\frac{2}{L}}\left(e^{-i\frac{n\pi x}{L}} - e^{i\frac{n\pi x}{L}}\right) \\
& \Phi(p) = \frac{-i}{\sqrt{2\pi\hbar}}\frac{1}{\sqrt{2L}}\int_0^L e^{-i\frac{px}{\hbar}}e^{i\frac{n\pi x}{L}}\mathrm{d}x + \frac{i}{\sqrt{2\pi\hbar}}\frac{1}{\sqrt{2L}}\int_0^L e^{-i\frac{px}{\hbar}}e^{-i\frac{n\pi x}{L}}\mathrm{d}x = \\
& = \frac{-i}{\sqrt{4\pi\hbar L}}\int_0^L e^{i\left(\frac{n\pi}{L}-\frac{p}{\hbar}\right)x}\mathrm{d}x + \frac{i}{\sqrt{4\pi\hbar L}}\int_0^L e^{-i\left(\frac{n\pi}{L}+\frac{p}{\hbar}\right)x}\mathrm{d}x = \\
& = \frac{-i}{\sqrt{4\pi\hbar L}}\frac{e^{i\left(\frac{n\pi}{L}-\frac{p}{\hbar}\right)L}-1}{i\left(\frac{n\pi}{L}-\frac{p}{\hbar}\right)} + \frac{i}{\sqrt{4\pi\hbar L}}\frac{e^{-i\left(\frac{n\pi}{L}+\frac{p}{\hbar}\right)L}-1}{-i\left(\frac{n\pi}{L}+\frac{p}{\hbar}\right)} = \\
& = \frac{-1}{\sqrt{4\pi\hbar L}}\frac{e^{-i\frac{pL}{\hbar}}-1}{\left(\frac{n\pi}{L}-\frac{p}{\hbar}\right)} - \frac{1}{\sqrt{4\pi\hbar L}}\frac{e^{-i\frac{pL}{\hbar}}-1}{\left(\frac{n\pi}{L}+\frac{p}{\hbar}\right)} = \\
& = \frac{-1}{\sqrt{4\pi\hbar L}}\left(e^{-i\frac{pL}{\hbar}}-1\right)\frac{2\frac{n\pi}{L}}{\left(\left(\frac{n\pi}{L}\right)^2-\left(\frac{p}{\hbar}\right)^2\right)} \qquad \blacksquare
\end{aligned}
$$

γ)

$$
\begin{aligned}
& |\Phi|^2 = \Phi^*(p)\Phi(p) = \frac{4\frac{n^2\pi^2}{L^2}}{4\pi\hbar L}\frac{\left(e^{i\frac{pL}{\hbar}}-1\right)\left(e^{-i\frac{pL}{\hbar}}-1\right)}{\left(\left(\frac{n\pi}{L}\right)^2-\left(\frac{p}{\hbar}\right)^2\right)^2} = \\
& = \frac{4n^2\pi}{\hbar L^3}\frac{\sin^2\left(\frac{pL}{2\hbar}\right)}{\left(\frac{n\pi}{L}-\frac{p}{\hbar}\right)^2\left(\frac{n\pi}{L}+\frac{p}{\hbar}\right)^2} \qquad \blacksquare
\end{aligned}
$$



Φιγυρε 1: Η γραφική παράσταση της πυκνότητας πιθανότητας συναρτήσει της ορμής για $\hbar = 1$, $L = 1$ και $n = 10$

Άσκηση 2.

α)

- $\langle x\rangle e = \int_{-\infty}^{+\infty} x\Psi^*(x)\Psi(x)\mathrm{d}x =$

$$= \sqrt{\frac{a}{\pi}}\int_{-\infty}^{+\infty} xe^{-a(x-x_0)^2}\mathrm{d}x =$$
$$= \sqrt{\frac{a}{\pi}}\int_{-\infty}^{+\infty} (y+x_0)\,e^{-ay^2}\mathrm{d}y =$$
$$= 0 + x_0\sqrt{\frac{a}{\pi}}\int_{-\infty}^{+\infty} e^{-ay^2}\mathrm{d}y = x_0\sqrt{\frac{a}{\pi}}\sqrt{\frac{\pi}{a}} = x_0$$

- $\langle x^2\rangle = \sqrt{\frac{a}{\pi}}\int_{-\infty}^{+\infty} x^2 e^{-a(x-x_0)^2}\mathrm{d}x = \sqrt{\frac{a}{\pi}}\int_{-\infty}^{+\infty}(y+x_0)^2 e^{-ay^2}\mathrm{d}x =$

$$= \sqrt{\frac{a}{\pi}}\left[\int_{-\infty}^{+\infty} y^2 e^{-ax^2}\mathrm{d}y + 2x_0\int_{-\infty}^{+\infty} ye^{-ay^2}\mathrm{d}y + x_0^2\int_{infty}^{+\infty} e^{-ay^2}\mathrm{d}y\right] =$$
$$= \sqrt{\frac{a}{\pi}}\left[\frac{1}{2a}\sqrt{\frac{\pi}{a}} + 0 + x_0^2\sqrt{\frac{\pi}{a}}\right] = x_0^2 + \frac{1}{2a}$$

- $\langle p\rangle = \int_{-\infty}^{+\infty}\Phi(x)e^{-i\frac{p_0 x}{\hbar}}(-i\hbar)\frac{\mathrm{d}\Psi}{\mathrm{d}x}\mathrm{d}x =$

$$= \int_{-\infty}^{+\infty}\Phi(x)e^{-i\frac{p_0 x}{\hbar}}(-i\hbar)\left[\frac{\mathrm{d}\Phi}{\mathrm{d}x}e^{i\frac{p_0 x}{\hbar}} + \left(\frac{ip_0}{\hbar}\right)\Phi e^{i\frac{p_0 x}{\hbar}}\right]\mathrm{d}x =$$

$$= (-i\hbar)\int_{-\infty}^{+\infty}\Phi(x)\frac{\mathrm{d}\Phi}{\mathrm{d}x}\mathrm{d}x + p_0\int_{-\infty}^{+\infty}\Phi^2\mathrm{d}x =$$

$$= \frac{-i\hbar}{2}\int_{-\infty}^{+\infty}\frac{\mathrm{d}\Phi^2}{\mathrm{d}x}\mathrm{d}x + p_0 = \frac{-i\hbar}{2}\Phi^2\Big|_{-\infty}^{+\infty} + p_0 = 0 + p_0 = p_0$$

- $\langle p^2\rangle = -\hbar^2\int_{-\infty}^{+\infty}\Psi^*(x)\Psi''(x)\mathrm{d}x =$

$$= -\hbar^2\int_{-\infty}^{+\infty}\frac{\mathrm{d}}{\mathrm{d}x}\left(\Psi^*\Psi'\right)\mathrm{d}x + \hbar^2\int_{-\infty}^{+\infty}\Psi'^*\Psi'\mathrm{d}x =$$

$$= -\hbar^2\Psi^*(x)\Psi'(x)\Big|_{-\infty}^{+\infty} + \hbar^2\int_{-\infty}^{+\infty}\Psi'^*\Psi'\mathrm{d}x$$

$$= \hbar^2\int_{-\infty}^{+\infty}\Psi'^*\Psi'\mathrm{d}x$$

$\Rightarrow$

$$\langle p^2\rangle = \hbar^2\int_{-\infty}^{+\infty}\left[\Phi'(x)e^{-i\frac{p_0x}{\hbar}} - \frac{ip_0}{\hbar}\Phi(x)e^{-i\frac{p_0x}{\hbar}}\right]\left[\Phi'(x)e^{i\frac{p_0x}{\hbar}} + \frac{ip_0}{\hbar}\Phi(x)e^{i\frac{p_0x}{\hbar}}\right]\mathrm{d}x =$$

$$= \hbar^2\int_{-\infty}^{+\infty}\left(\Phi'\right)^2\mathrm{d}x - ip_0\hbar\int_{-\infty}^{+\infty}\Phi\Phi'\mathrm{d}x + ip_0\hbar\int_{-\infty}^{+\infty}\Phi'\Phi\mathrm{d}x + p_0^2\int_{-\infty}^{+\infty}\Phi^2\mathrm{d}x =$$

$$= p_0^2 + \hbar^2\sqrt{\frac{a}{\pi}}\int_{-\infty}^{+\infty}a^2\left(x-x_0\right)^2 e^{-a(x-x_0)^2}\mathrm{d}x =$$

$$= p_0^2 + a^2\hbar^2\left(x_0^2 + \frac{1}{2a} - 2x_0^2 + x_0^2\right) =$$

$$= p_0^2 + \hbar^2\frac{a}{2} \qquad \blacksquare$$

β)

- $(\Delta x)^2 = \langle x^2\rangle - \langle x\rangle^2 = x_0^2 + \dfrac{1}{2a} - x_0^2 = \dfrac{1}{2a}$
- $(\Delta p)^2 = \langle p^2\rangle - \langle p\rangle^2 = p_0^2 + \hbar^2\dfrac{a}{2} - p_0^2 = \hbar^2\dfrac{a}{2}$
- $(\Delta x)\,(\Delta p) = \dfrac{1}{\sqrt{2a}}\sqrt{\dfrac{a}{2}}\hbar = \dfrac{\hbar}{2} \qquad \blacksquare$

Άσκηση 3.

Ορίζουμε πρώτα την κυματοσυνάρτηση στον χώρο των ορμών

$$\Phi(p) = \int_{-\infty}^{+\infty} \Psi_p^*(x)\Psi(x,0)\mathrm{d}x = \frac{1}{\sqrt{2\pi\hbar}}\left(\frac{\lambda}{\pi}\right)^{\frac{1}{4}} \int_{-\infty}^{+\infty} e^{-\lambda\frac{x^2}{2}} e^{-i\frac{px}{\hbar}}\mathrm{d}x =$$
$$= \frac{1}{\sqrt{2\pi\hbar}}\left(\frac{\lambda}{\pi}\right)^{\frac{1}{4}} \sqrt{\frac{2\pi}{\lambda}} e^{-\frac{p^2}{2\lambda\hbar^2}} = \left(\frac{1}{\lambda\pi\hbar^2}\right)^{\frac{1}{4}} e^{-\frac{p^2}{2\lambda\hbar^2}}$$

Έχουμε:

$$\int_{-\infty}^{+\infty} e^{-\alpha x^2+\beta x}\mathrm{d}x = \sqrt{\frac{\pi}{\alpha}} e^{\frac{\beta^2}{4\alpha}}$$

με $\mathrm{Re}(\alpha) > 0$ και $\beta =$ οτιδήποτε
Ορίζουμε την $\Psi(x,t)$

$$\Psi(x,t) = \int_{-\infty}^{+\infty} \Phi(p)\Psi_p(x)e^{-i\frac{E_p t}{\hbar}}\mathrm{d}p =$$
$$= \left(\frac{1}{\lambda\pi\hbar^2}\right)^{\frac{1}{4}} \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} e^{-\frac{p^2}{2\lambda\hbar^2}} e^{i\frac{px}{\hbar}} e^{-i\frac{p^2 t}{2m\hbar}}\mathrm{d}p =$$
$$= \frac{1}{\sqrt{2\pi\hbar}} \frac{1}{(\lambda\pi\hbar^2)^{\frac{1}{4}}} \int_{-\infty}^{+\infty} e^{-\;\;\frac{1}{2\lambda\hbar^2}+\frac{it}{2m\hbar}\;\; p^2+\frac{ix}{\hbar}p}\mathrm{d}p =$$
$$= \frac{1}{\sqrt{2\pi\hbar}} \frac{1}{(\lambda\pi\hbar^2)^{\frac{1}{4}}} \sqrt{\frac{\pi}{\left(\frac{1}{2\lambda\hbar^2}+\frac{it}{2m\hbar}\right)}}\, e^{\left(\frac{ix}{\hbar}\right)^2 \frac{1}{4\left(\frac{1}{2\lambda\hbar^2}+\frac{it}{2m\hbar}\right)}} =$$
$$= N(t)e^{-\frac{\lambda(t)}{2}x^2}$$

όπου

$$N(t) = \frac{1}{\sqrt{2\pi\hbar}} \frac{1}{(\lambda\pi\hbar^2)^{\frac{1}{4}}} \sqrt{\frac{\pi}{\left(\frac{1}{2\lambda\hbar^2}+\frac{it}{2m\hbar}\right)}}$$

και

$$\lambda(t) = \frac{1}{2\hbar^2}\frac{1}{\frac{1}{2\lambda\hbar^2}+\frac{it}{2m\hbar}} = \frac{1}{2\hbar^2}\frac{4\lambda m\hbar^3}{2m\hbar+i2\lambda t\hbar^2}$$
$$= \frac{\lambda m}{m+i\lambda t\hbar} = \frac{\lambda m(m-i\lambda t\hbar)}{m^2+\lambda^2t^2\hbar^2} = \lambda_1(t) - i\lambda_2(t)$$

με

$$\begin{aligned}
\lambda_1(t) &= \frac{\lambda m^2}{m^2+\lambda^2 t^2 \hbar 2} \\
\lambda_2(t) &= \frac{\lambda^2 t \hbar m}{m^2+\lambda^2 t^2 \hbar^2}
\end{aligned}$$

Άρα

$$\Psi(x,t) = N(t) e^{i\frac{\lambda_2(t)}{2}x^2} e^{-\frac{\lambda_1(t)x^2}{2}}$$

όπου $\lambda_1(t) \to 0$ για $t \to \infty$
Επίσης

$$\begin{aligned}
P(x,t) &= |\Psi(x,t)|^2 = |N(t)|^2 e^{-\lambda_1 x^2} \\
|N(t)|^2 &= \sqrt{\frac{\lambda_1(t)}{\pi}}
\end{aligned}$$

και

$$\begin{aligned}
(\Delta x)_t &= \frac{1}{\sqrt{2\lambda_1(t)}} = \frac{1}{\sqrt{2\lambda}}\sqrt{1+\frac{\lambda^2\hbar^2 t^2}{m^2}} \xrightarrow[t\to\infty]{} \infty \\
(\Delta x)_{t=0} &= \frac{1}{\sqrt{2\lambda}}
\end{aligned}$$

Άρα το κυματοπακέτο απλώνει με τον χρόνο και η πιθανότητα $P(x,t) \to 0$ για $t \to \infty$. ■

Άσκηση 4.

$$\begin{aligned}
-\frac{\hbar^2}{2m}\nabla^2\Psi + V_1\Psi + iV_2\Psi &= i\hbar\frac{\partial\Psi}{\partial t} \\
-\frac{\hbar^2}{2m}\nabla^2\Psi^* + V_1\Psi^* - iV_2\Psi^* &= -i\hbar\frac{\partial\Psi^*}{\partial t}
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial(\Psi^*\Psi)}{\partial t} = \frac{\partial\Psi^*}{\partial t}\Psi + \Psi^*\frac{\partial\Psi}{\partial t} = \\
&= \frac{\hbar}{2im}\Psi\nabla^2\Psi^* - \frac{1}{i\hbar}V_1\Psi^*\Psi + \frac{iV_2}{i\hbar}\Psi^*\Psi - \frac{\hbar}{2im}\Psi^*\nabla^2\Psi + \frac{1}{i\hbar}V_1\Psi^*\Psi + \frac{iV_2}{i\hbar}\Psi^*\Psi = \\
&= -\frac{\hbar}{2im}\left(\Psi^*\nabla^2\Psi - \Psi\nabla^2\Psi^*\right) + \frac{2}{\hbar}V_2\Psi^*\Psi = \\
&= -\frac{\hbar}{2im}\nabla\cdot\left(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*\right) + \frac{2}{\hbar}V_2\Psi^*\Psi
\end{aligned}$$

Θεωρώντας $P=\Psi^*\Psi$ και $\mathbf{J}=\frac{\hbar}{2im}\left(\Psi^*\nabla\Psi-\Psi\nabla\Psi^*\right)$ παίρνουμε

$$\frac{\partial P}{\partial t}+\nabla\cdot\mathbf{J}=\frac{2}{\hbar}V_2P \qquad \blacksquare$$

Άσκηση 6.

α) Ισχύει $[l_x,l_y]=i\hbar l_z$, $[l_z,l_x]=i\hbar l_y$, και $[l_z,l_y]=-i\hbar l_x$, άρα

$$\begin{aligned}&[l_z,l_+]=[l_z,l_x+il_y]=[l_z,l_x]+i\,[l_z,l_y]=\\&=i\hbar l_y+i(-i\hbar)l_x=\hbar\,(l_x+il_y)=\hbar l_+ \qquad \blacksquare\end{aligned}$$

β)

$$\begin{aligned}&l_z\Phi=l_zl_+\Psi=l_+l_z\Psi+\hbar l_+\Psi=\\&=l_+(\lambda\Psi)+\hbar l_+\Psi=(\lambda+\hbar)l_+\Psi\end{aligned}$$

διότι $l_zl_+-l_+l_z=\hbar l_+$
και $l_zl_+=l_+l_z+\hbar l_+$ $\blacksquare$

Άσκηση 7.

β)

$$\begin{aligned}&\langle\Psi_n,\mathbf{P}\Psi_n\rangle=\frac{im}{\hbar}\langle\Psi_n\,[H,\mathbf{r}]\,\Psi_n\rangle=\\&=\frac{im}{\hbar}\left[\langle\Psi_n,H\mathbf{r}\Psi_n\rangle-\langle\Psi_n,\mathbf{r}H\Psi_n\rangle\right]=\\&=\frac{im}{\hbar}\left[E_n\langle\Psi_n,\mathbf{r}\Psi\rangle-E_n\langle\Psi_n,\mathbf{r}\Psi_n\rangle\right]=0 \qquad \blacksquare\end{aligned}$$

γ)

$$\begin{aligned}&\langle\Psi_n,H\mathbf{r}\Psi_n\rangle=\int\Psi_n^*H(\mathbf{r}\Psi_n)\mathrm{d}^3x=\\&=\int(H\Psi_n)^*\,(\mathbf{r}\Psi_n)\,\mathrm{d}^3x=E_n\int\Psi^*\,(\mathbf{r}\Psi_n)\,\mathrm{d}^3x \qquad \blacksquare\end{aligned}$$

Άσκηση 8.

Καταρχήν $F = F(A,x)$ και $[p_x, A] = 0$
Εάν θέλουμε να ορίσουμε την συνάρτηση $g(\hat{B})$ ενός τελεστή $\hat{B}$ θεωρούμε την ανάλυση της $g(x)$ σε σειρά Taylor γύρω από το σημείο $x = 0$:

$$g(x) = \sum_{n=0}^{\infty} \frac{1}{n!} g^{(n)}(x=0) x^n$$
$$\Rightarrow g(\hat{B}) = \sum_{n=0}^{\infty} \frac{1}{n!} g^{(n)}(0) \hat{B}^n$$

Εάν έχουμε την φυσική ποσότητα $C = xp$ τότε ο τελεστής $\hat{C}$ είναι ερμιτιανός εάν ορίσουμε συμμετρικά ως προς τα $\hat{x}$,$\hat{p}$

$$\hat{C} = \frac{1}{2}\left(\hat{x}\hat{p} + \hat{p}\hat{x}\right)$$

Ορίζουμε λοιπόν την ποσότητα $F(A,x)$ ως εξής:

$$F(\hat{A},\hat{x}) = \frac{1}{2}\left[\sum_{n=0}^{\infty} \frac{1}{n!}\left[F^{(n)}(A,0)x^n + x^n F^{(n)}(A,0)\right]\right]$$

όπου

$$F^{(n)}(A,0) = \left.\frac{\mathrm{d}^n F(A,x)}{\mathrm{d}x^n}\right|_{x=0}$$

Επομένως

$$[p_x, F] = \left[p_x, \sum_{n=0}^{\infty} \frac{1}{n!} F^{(n)}(A,0) x^n\right] =$$
$$= \sum_{n=0}^{\infty} \frac{1}{n!}\left[\left[p_x, F^{(n)}(A,0)\right] x^n + F^{(n)}(A,0)\left[p_x, x^n\right]\right] =$$
$$= 0 + \sum_{n=0}^{\infty} \frac{1}{n!} F^{(n)}(A,0)(-i\hbar)\frac{\partial x^n}{\partial x} =$$
$$= (-i\hbar)\frac{\partial}{\partial x}\left[\sum_{n=0}^{\infty} \frac{1}{n!} F^{(n)}(A,0) x^n\right] = (-i\hbar)\frac{\partial F}{\partial x} \qquad \blacksquare$$

Άσκηση 9.

Δοκιμάζουμε το ολοκλήρωμα με μια τυχούσα συνάρτηση $f(x)$

$$\int_{-\infty}^{+\infty} f(x)\frac{\mathrm{d}\zeta}{\mathrm{d}x}\mathrm{d}x = \int_{-\infty}^{+\infty} \frac{\mathrm{d}}{\mathrm{d}x}(f\zeta)\mathrm{d}x - \int_{-\infty}^{+\infty} \zeta(x)\frac{\mathrm{d}f}{\mathrm{d}x}\mathrm{d}x =$$
$$= [f(+\infty)\zeta(+\infty) - f(-\infty)\zeta(-\infty)] - \int_{-\infty}^{0} (-a)\frac{\mathrm{d}f}{\mathrm{d}x}\mathrm{d}x - \int_{0}^{+\infty} a\frac{\mathrm{d}f}{\mathrm{d}x}\mathrm{d}x =$$
$$= [af(+\infty) - (-a)f(-\infty)] + a\,[f(0) - f(-\infty)] - a\,[f(+\infty) - f(0)] =$$
$$= af(+\infty) - af(+\infty) + af(-\infty) - af(-\infty) + 2af(0) =$$
$$= 2af(0) = 2a\int_{-\infty}^{+\infty} f(x)\delta(x)\mathrm{d}x \qquad \blacksquare$$

Άσκηση 10.

α)

$$e^{-sA} = \sum_{n=0}^{\infty}\frac{(-sA)^n}{n!} = \sum_{n=0}^{\infty}\frac{(-1)^n s^n}{n!}A^n$$
$$\left[e^{-sA}, B\right] = \sum_{n=0}^{\infty}\frac{(-1)^n s^n}{n!}\,[A^n, B]$$
$$[A^n, B] = n\,[A, B]\,A^{n-1}$$
$$\Rightarrow \left[e^{-sA}, B\right] = \sum_{n=0}^{\infty}\frac{(-1)^n s^n}{n!}\,[A, B]\,nA^{n-1} =$$
$$= (-s)\,[A, B]\sum_{n=1}^{\infty}\frac{(-1)^{n-1}s^{n-1}}{(n-1)!}A^{n-1} = (-s)\,[A, B]\,e^{-sA} \qquad \blacksquare$$

β)

$$e^{-sA}Be^{sA} = e^{-sA}Be^{sA} + Be^{-sA}e^{sA} - Be^{-sA}e^{sA} =$$
$$= \left[e^{-sA}, B\right]e^{sA} + Be^{-sA}e^{sA} =$$
$$= -s\,[A, B]\,e^{-sA}e^{sA} + Be^{-sA}e^{sA} = -s\,[A, B] + B$$

$$e^{-sA}e^{sA} = \left[1 - sA + \frac{(sA)^2}{2!} - \frac{(sA)^3}{3!} + \cdots\right]\left[1 + sA + \frac{(sA)^2}{2!} + \frac{(sA)^3}{3!} + \cdots\right] =$$
$$= 1 + (sA - sA) + \left[\frac{(sA)^2}{2} - (sA)^2 + \frac{(sA)^2}{2}\right] + s^3\,[\cdots] + \cdots =$$
$$= 1 + 0 + 0 + 0 + \cdots = 1 \qquad \blacksquare$$

Άσκηση 11.

α)

$$
\begin{aligned}
&A = A(\omega)\\
&\frac{\mathrm{d}A}{\mathrm{d}\omega} = \lim_{\Delta\omega\to 0}\frac{A(\omega+\Delta\omega)-A(\omega)}{\Delta\omega} \qquad \blacksquare
\end{aligned}
$$

β)

$$
\begin{aligned}
&\frac{\mathrm{d}(AB)}{\mathrm{d}\omega} = \lim_{\Delta\omega\to 0}\frac{A(\omega+\Delta\omega)B(\omega+\Delta\omega)-A(\omega)B(\omega)}{\Delta\omega} =\\
&= \lim_{\Delta\omega\to 0}\frac{1}{\Delta\omega}\left[A(\omega+\Delta\omega)B(\omega+\Delta\omega)+A(\omega)B(\omega+\Delta\omega)-A(\omega)B(\omega+\Delta\omega)-A(\omega)B(\omega)\right] =\\
&= \lim_{\Delta\omega\to 0}\frac{1}{\Delta\omega}\left[A(\omega+\Delta\omega)-A(\omega)\right]B(\omega+\Delta\omega)+\lim_{\Delta\omega\to 0}\frac{A(\omega)}{\Delta\omega}\left[B(\omega+\Delta\omega)-B(\omega)\right] =\\
&= \frac{\mathrm{d}A}{\mathrm{d}\omega}B + A\frac{\mathrm{d}B}{\mathrm{d}\omega} \qquad \blacksquare
\end{aligned}
$$

γ)

$$
\begin{aligned}
&AA^{-1} = 1 \Rightarrow \frac{\mathrm{d}}{\mathrm{d}\omega}(AA^{-1}) = 0\\
&\Rightarrow \frac{\mathrm{d}A}{\mathrm{d}\omega}A^{-1} + A\frac{\mathrm{d}A^{-1}}{\mathrm{d}\omega} = 0\\
&\Rightarrow \frac{\mathrm{d}A}{\mathrm{d}\omega}A^{-1} = -A\frac{\mathrm{d}A^{-1}}{\mathrm{d}\omega} \Rightarrow \frac{\mathrm{d}A^{-1}}{\mathrm{d}\omega} = -A^{-1}\frac{\mathrm{d}A}{\mathrm{d}\omega}A^{-1}\\
&\Rightarrow \frac{\mathrm{d}A}{\mathrm{d}\omega} = -A\frac{\mathrm{d}A^{-1}}{\mathrm{d}\omega}A \qquad \blacksquare
\end{aligned}
$$