

15/9/10 Θερμού (Sturm-Liouville)

a) Να λογισθεί οι σιωπής και οι αντίστοιχες σιωπηρές του πρόβληματος ευρωπαϊκής γήρας:

$$x^2 y''(x) + x y'(x) + \lambda y(x) = 0, \quad 1 < x < e^n, \quad y(1) = y(e^n) = 0$$

$$\text{Θερμ } x = e^t \text{ οπότε: } \frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx} = y'(t) \frac{1}{e^t} = y'(t) \frac{1}{e^t} = y'(t) e^{-t}$$

$$\frac{d^2y}{dx^2} = \frac{d(y'(t)e^{-t})}{dt} \frac{dt}{dx} = (y''(t)e^{-t} - y'(t)e^{-t}) e^{-t} = (y''(t) - y'(t)) e^{-2t}$$

$$\text{Οπότε το πρόβλημα γίνεται: } e^{2t} e^{-2t} (y''(t) - y'(t)) + e^t e^{-t} y'(t) + \lambda y(t) = 0 \Rightarrow \\ y''(t) - y'(t) + y'(t) + \lambda y(t) = 0 \Rightarrow y''(t) + \lambda y(t) = 0 \quad \text{και}$$

αν $x \geq 1 \Rightarrow e^t \geq 1 \Rightarrow t \geq 0$. Αν $x \leq e^n \Rightarrow e^t \leq e^n \Rightarrow t \leq n$ και $y(0) = 0, y(n) = 0$

- Αν $\lambda = 0 \Rightarrow y''(t) = 0 \Rightarrow y'(t) = C_1 \Rightarrow y(t) = C_1 t + C_2$. Στα $t=0 \Rightarrow y(0)=0 \Rightarrow C_2=0$. Στα $t=n \Rightarrow y(n)=0 \Rightarrow 0 = C_1 n \Rightarrow C_1=0$ (Από $\lambda=0$ δεν γίνεται σιωπή)

$$\bullet \quad \begin{aligned} &\text{Αν } \lambda = -k^2 < 0 \text{ τότε } y''(t) - k^2 y(t) = 0 \text{ άρα } y(t) = A e^{kt} + B e^{-kt} \\ &y(0) = 0 \quad \left\{ \begin{array}{l} A+B=0 \\ A e^{n k} + B e^{-n k} = 0 \end{array} \right\} \quad \left\{ \begin{array}{l} B=-A \\ A(e^{n k} - e^{-n k}) = 0 \end{array} \right\} \quad \left\{ \begin{array}{l} A=0 \\ B=0 \end{array} \right. \end{aligned}$$

Άρα δεν υπάρχει απρωτής σιωπής.

- Av $\lambda = k^2 \geq 0$ zor $y(t) = A \sin kt + B \cos kt$

Tie $t=0$ s.kw $y(0)=0 \Rightarrow B=0$

Tie $t=\pi$ s.kw $y(\pi)=0 \Rightarrow A \sin k\pi=0 \Rightarrow k\pi=n\pi \Rightarrow k_n=n, n=1, \dots$

öpp. zo npöblmje exi. lösung je $\lambda_n = n^2$ van idiosyncrasies
 $y_n = A_n \sin(nt), n=1, 2, \dots$

b) Eiven n lösungen ges Sturm-Liouvillej N. d'wirre
 zu exakt opförmiches nor rörelsern oso zo npöblmje

$P(x)y''(x) + Q(x)y'(x) + \lambda y(x) = 0$ van npöblm $P'(x) = Q(x)$

sd i. $P(x) = 1 \Rightarrow P'(x) \geq 0$ van $Q(x) = 0$ oso tie n lösure

öpp. n lösungen $y''(x) + \lambda y(x) = 0$ eiven ges Sturm-Liouville.

enior $y_1(x) - y_2(x) + y_1(x) \cdot 0 \cdot y_2'(x) + \lambda y_1(x) y_2(x) = 0$ van npöblm
 $y_1'(x) = 0 \cdot y_2(x) \Rightarrow y_1'(x) = C$ van enidspunkt $C=1$ öpp. $y_1(x) = 1$

Hörer opförmiches (j.e. y_n, y_m für $n \neq m$ lös idiosyncrasies

zo rörelsern) eiven:

$\int_0^\pi y_n(x) A_n \sin(nx) A_m \sin(mx) dt = 0 \Rightarrow$

$\int_0^\pi A_n \sin(nx) A_m \sin(mx) dt = 0$

j) Ne huvd, j.s. en lösning av enersens ge nöpplmje
 idiosyncrasies (svallarum lösningar Fredholm) zo uppgiftens npöblmje
 $x^2 y''(x) + x y'(x) + 3 y(x) = 0$, $L(x)e^x, y(1) = y(e^{\frac{1}{2}}) = 0$

(sd i. ex. $\lambda = 3$ van $g(x) = 1$. Av vare m x zor $y(x) = e^{xt}$)

- $y(x) = \sum_{n=1}^{\infty} a_n y_n(x)$

(i) Av $\lambda - \lambda_n \neq 0, \forall n \exists$ lösning lös

- $a_n = \frac{I_n}{\|y_n\|_n^2}$

(ii) Av $\lambda - \lambda_n = 0$ van $I_n \neq 0$ adjunto

- $I_n = \int_0^\pi g(x) y_n(x) dt$

(iii) Av $\lambda - \lambda_n = 0$ van $I_n = 0$ eöp. 1670 van

n lös lösung ges Sturm-Liouville:

- $\|y_n\|_n^2 = \int_0^\pi y_n(x)^2 dx$

$y(x) = \sum_{n=1}^{\infty} a_n y_n(x) + C y_n(x), n \neq m.$

$\|y_n(x)\|_n^2 = \int_0^\pi \sin^2 nx dt = \frac{\pi}{2}$

$I_n = \int_0^\pi \sin(nx) dt = -\frac{1}{n} \int_0^\pi (\cos nt)' dt = -\frac{1}{n} \cos nt \Big|_0^\pi = -\frac{1}{n} (\cos n\pi - 1) =$

$= -\frac{1}{n} [(-1)^n - 1]$

öpp. $a_n = \frac{-\frac{1}{n} [(-1)^n - 1]}{(3-n^2) \frac{\pi}{2}} = -\frac{2[(-1)^n - 1]}{(3-n^2) n \pi}$

$y(x) = \sum_{n=1}^{\infty} -\frac{2[(-1)^n - 1]}{(3-n^2) n \pi} \cdot \sin(nx)$

δ) Ήμε αναπτύξτε τη συνάριθμη υπό Fourier, με ρος της
διαδικασίας των (e), τη διαδικασία $y(x)=1$, $0 < x < L$.

$$y(x)=1 = \sum_{n=1}^{\infty} (\lambda \rightarrow n) a_n y_n(t) \Rightarrow \sum_{n=1}^{\infty} (3-n^2) \left(-\frac{2(-2)^n - 1}{(3-n^2)n\pi} \right) \sin(nt) = 1$$

$$\Rightarrow \sum_{n=1}^{\infty} -\frac{2(-2)^n - 1}{nn} \sin(nt) = 1$$

18/6/10 Εφετές 4^o (Sturm-Liouville)

a) Η αρχή για τη διαδικασία της αναπτύξεως διαδικασίας
των προβλημάτων ευρετήριων τύπων:

$$y''(x) + 2y'(x) + \lambda y(x) = 0, \quad 0 < x < L, \quad y(0) = y(L) = 0$$

$$\lambda^2 - 2\lambda + 4 = 0 \Rightarrow \Delta = 4 - 4\lambda = 4(1-\lambda) \Rightarrow \lambda_{1,2} = \frac{+2 \pm \sqrt{4(1-\lambda)}}{2} = 1 \pm \sqrt{1-\lambda}$$

- Για $\lambda = L$, $\Delta = 0$, $\lambda_1 = \lambda_2 = 1$

$$y(x) = A e^{kx} + B x e^{kx} \quad \text{για } y(0) = 0 \Rightarrow A = 0 \text{ και } y(L) = 0 \Rightarrow B L e^L = 0 \Rightarrow B = 0$$

επομένως αναπτύξεως των $\lambda = 1$

- Για $\lambda < L$, $\Delta > 0$ αριθμοί k_1, k_2 με $k_1 \neq k_2$

$$y(x) = A e^{k_1 x} + B e^{k_2 x}$$

$$\begin{cases} y(0) = A + B = 0 \\ y(L) = A e^{k_1 L} + B e^{k_2 L} = 0 \end{cases} \Rightarrow \begin{cases} A + B = 0 \\ A e^{k_1 L} - A e^{k_2 L} = 0 \end{cases} \Rightarrow \begin{cases} A = -B \\ A(e^{k_1 L} - e^{k_2 L}) = 0 \end{cases} \Rightarrow \begin{cases} A = -B \\ A = 0 \text{ ή } k_1 = k_2 \end{cases}$$

επειδή αναπτύξεως της τύπου $\lambda < L$

- Για $\lambda > L$, $\Delta < 0$ $\lambda_1 = 1 + i\sqrt{\lambda-1}$, $\lambda_2 = 1 - i\sqrt{\lambda-1}$

$$y(x) = A e^{k_1 x} \cos(\sqrt{\lambda-1} x) + B e^{k_1 x} \sin(\sqrt{\lambda-1} x) \quad \text{για } y(0) = 0 \Rightarrow A = 0$$

$$\text{και } y(L) = 0 \Rightarrow e^{k_1 L} \sin(\sqrt{\lambda-1} L) = 0 \Rightarrow \sqrt{\lambda-1} L = n\pi \Rightarrow \lambda_n = \frac{n^2\pi^2}{L^2} + L \quad n=1, 2, \dots$$

$$\text{επειδή } y_n(x) = B_n e^{k_1 x} \sin\frac{n\pi}{L} x, \quad n=1, 2, \dots$$

(b) Een van de oplossingen van Sturm-Liouville; Noch een
in evenwichtige toestand. De waarden van de eigenwaarden voor
spoorwaarden en opeenvolgende zijn.

$$P(x)y''(x) + Q(x)y'(x) + \lambda y(x) = 0 \text{ van waar } P'(x) = Q(x)$$

$$\text{dus } P(x) = 1 \Rightarrow P'(x) = 0 \text{ van waar } Q(x) = -2 \text{ op de vorm}$$

$$\text{van de evenwichtige toestand: } y(x) - 2y'(x) + \lambda y(x) = 0 \\ \text{van waar } y'(x) = -2y(x) \Rightarrow y(x) = C \cdot e^{-2x}. \text{ En dan: } C=1 \text{ op de} \\ y(x) = e^{-2x}$$

$$\text{van de eigenwaarden voor: } \int_0^L y(x) y_n(x) dx = 0 \text{ voor } n \neq m \Rightarrow \\ \Rightarrow \int_0^L e^{-2x} \cdot e^x \sin \frac{n\pi x}{L} \cdot e^x \sin \frac{m\pi x}{L} dx = 0 \Rightarrow \int_0^L \sin \left(\frac{n\pi x}{L} \right) \sin \left(\frac{m\pi x}{L} \right) dx = 0$$

j) Neemt nu de vorm van de oplossing van de evenwichtige toestand
(evaluatie van Fredholm), zo dat het probleem:

$$y''(x) - 2y'(x) + 2y(x) = e^x \quad 0 < x < L, \quad y(0) = y(L) = 0$$

$$y(x) = \sum_{n=1}^{\infty} a_n y_n(x), \quad a_n = \frac{1}{(2\pi n)} \|y_n(x)\|_p^2 \quad I_n = \int_0^L e^x y_n(x) dx$$

$$\|y_n(x)\|_p^2 = \int_0^L y_n^2(x) dx$$

$$\left. \begin{array}{l} \text{(i) Av } 2-\lambda n \neq 0 \text{ van } I_n \text{ voor elke } n \\ \text{(ii) Av } 2-\lambda n = 0 \text{ van } I_m \neq 0 \text{ anders} \\ \text{(iii) Av } 2-\lambda n = 0 \text{ van } I_m = 0 \text{ oplegt een gedrag van de vorm } \end{array} \right\} \text{oplossing}$$

$$y(x) = \sum_{n=1}^{\infty} a_n y_n(x) + C y_m(x), \quad n \neq m$$

$$\|y_n(x)\|_p^2 = \int_0^L \left(e^x \sin \frac{n\pi x}{L} \right)^2 e^{-2x} dx = \int_0^L \sin^2 \left(\frac{n\pi}{L} x \right) dx =$$

$$= \int_0^L 1 - \cos^2 \left(\frac{n\pi}{L} x \right) dx = \int_0^L 1 - \frac{1}{2} - \cos \left(\frac{2n\pi}{L} x \right) dx =$$

$$= \int_0^L \frac{1}{2} dx - \frac{1}{2} \int_0^L \cos \left(\frac{2n\pi}{L} x \right) dx = \frac{L}{2} - \left[\frac{1}{2} \frac{L}{2n\pi} \sin \left(\frac{2n\pi}{L} x \right) \right]_0^L =$$

$$= \frac{L}{2} - \frac{L}{4n\pi} (\sin(2n\pi) - \sin(0)) = \frac{L}{2}$$

$$I_n = \int_0^L e^{2x} \sin \frac{n\pi x}{L} dx = -\frac{L}{n\pi} \int_0^L e^{2x} (\cos \frac{n\pi x}{L})' dx =$$

$$= -\frac{L}{n\pi} \left[e^{2x} \cos \frac{n\pi x}{L} \right]_0^L + \frac{L}{n\pi} \int_0^L 2e^{2x} \cos \frac{n\pi x}{L} dx =$$

$$= -\frac{L}{n\pi} \left[e^{2L} \cos n\pi - 1 \right] + \frac{2L^2}{n^2\pi^2} \int_0^L e^{2x} (\sin \frac{n\pi x}{L})' dx =$$

$$= -\frac{L}{n\pi} e^{2L} (-1)^n + \frac{L}{n\pi} + \left[\frac{2L^2}{n^2\pi^2} e^{2x} \sin \frac{n\pi x}{L} \right]_0^L - \frac{2L^2}{n^2\pi^2} \int_0^L 2e^{2x} \sin \frac{n\pi x}{L} dx =$$

$$= -\frac{L}{n\pi} e^{2L} (-1)^n + \frac{L}{n\pi} - \frac{4L^2}{n^2\pi^2} I_n \Rightarrow I_n = \frac{n\pi L - n\pi L e^{2L} (-1)^n}{n^2\pi^2 + 4L^2}$$

$$\text{οπόις: } \theta_{nL} = \frac{\frac{n\pi L - n\pi L e^{iL}}{(n^2\pi^2 + 4L^2)}}{2} = \frac{2n\pi L e^{iL} (-1)^n}{(2-n)L(n^2\pi^2 + 4L^2)} \text{ μακριά}$$

$y(x) = \sum_{n=1}^{\infty} a_n y_n(x)$ με $2 - \lambda n \neq 0$ μέρην παρουσίαν
 $(\lambda n = \frac{n^2\pi^2}{L^2} + 1 \neq 2)$

δ) Η επέκτηση της γενικήσης αριθμού Fourier (με ρησ της ιδεογραφίας του προβλήματος), τη γενικήση $g(x)$, για OXL.

$$g(x) = e^x = \sum_{n=1}^{\infty} (2 - \lambda n) a_n y_n(x) y_n(x)$$

Kατηγορίες για την διεργασία (για προσδιόριση 2nd γένους)

- Γραμμικός: Είναι τας γραμμικός

$$a(x)u_{xx} + b(x)u_{x_1x_2} + c(x)u_{x_2x_1} + d(x)u_{x_2x_2} + e(x)u_{x_1} + f(x)u_{x_2} + g(x)u = h(x)$$

- Μη γραμμικός:

$$a(x)u_{xx} + b(x)u_{x_1x_2} + c(x)u_{x_2x_1} + d(x)u_{x_2x_2} + e(x,u,\nabla u) = 0$$

για $\nabla u = (u_{x_1}, u_{x_2})$. Εδώ ουσιανή θεωρία στην εργασία των μεταβολισμών και σεπαραντών για την επέκτηση παραπάνω παραβολής.

- Εξοδικός:

$$a(x,u,\nabla u)u_{xx} + b(x,u,\nabla u)u_{x_1x_2} + c(x,u,\nabla u)u_{x_2x_1} + d(x,u,\nabla u)u_{x_2x_2} + e(x,u,\nabla u) = 0$$

Αν n ΜΔΕ είναι • Γραμμικός • 2nd γένους • Διάλογος παραβολής της γραμμικής παραβολής $u(x,t)$, στην γενική $u(x,t)$

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_{xt} + Eu_{yt} + Fu = G \text{ με λαντρώ } \Delta = \beta^2 - 4AC$$

- Αν $\Delta > 0$: Ουσιαστικοί ρίζοι
- Αν $\Delta < 0$: Ελλειπτικοί ρίζοι
- Αν $\Delta = 0$: Γερακολημένοι ρίζοι

Τεριέ για την κατηγοριοποίηση της ΜΔΕ λατρεί: ① Τάξη ② Από προσεπικόν παραβολή ③ Πεττιώντας ④ Οιολογίσμα (αν είναι γραμμικός, 2nd γέν.) ⑤ Τίνος (αν είναι γραμμικός, 2nd γέν.)

9/06 $\Theta_2 \geq 2^\circ$ (Klasseur)

Na $\sum \partial_i$ zo op te lezen opeenvolgende waarden van x en y

$$(1) \quad u_{tt}(x,y,t) = c^2 [u_{xx}(x,y,t) + u_{yy}(x,y,t)] \quad 0 < x < L, \quad 0 < y < 2, \quad t > 0$$

$$u(0,y,t) = u(L,y,t) = 0, \quad 0 < y < 2, \quad t > 0$$

$$u(x,0,t) = u(x,2,t) = 0, \quad 0 < x < L, \quad t > 0$$

$$u(x,y,0) = 3 \sin 2\pi x \sin 3\pi y, \quad 0 < x < L, \quad 0 < y < 2$$

$$u_t(x,y,0) = 0, \quad 0 < x < L, \quad 0 < y < 2$$

$$(1) \Rightarrow u_{xx}(x,y,t) + u_{yy}(x,y,t) - \frac{1}{c^2} u_{tt} = 0$$

Oefening 1 van de oppervlak $u(x,y,t) = X(x)Y(y)T(t)$ op de v(1) voorwaarde:

Gegeven dat $X(x)Y(y)T(t)$

$$X''(x)Y(y)T(t) + X(x)Y''(y)T(t) - \frac{1}{c^2} X(x)Y(y)T''(t) = 0 \Rightarrow$$

$$\Rightarrow \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} - \frac{1}{c^2} \frac{T''(t)}{T(t)} = 0 \text{ van } \Theta \text{ is } w$$

$$\frac{X''(x)}{X(x)} = k_1, \quad \frac{Y''(y)}{Y(y)} = k_2, \quad -\frac{1}{c^2} \frac{T''(t)}{T(t)} = k_3 \quad \text{v.s. } k_1 + k_2 + k_3 = 0$$

$$\bullet X''(x) - k_1 X(x) = 0 \stackrel{k_1 = -b^2}{\Rightarrow} X(x) = A_1 \cos b_1 x + B_1 \sin b_1 x$$

$$\bullet Y''(y) - k_2 Y(y) = 0 \stackrel{k_2 = -b_2^2}{\Rightarrow} Y(y) = A_2 \cos b_2 y + B_2 \sin b_2 y$$

$$\bullet T''(t) + \frac{1}{c^2} k_3 T(t) = 0 \Rightarrow T''(t) + c^2(b_1^2 + b_2^2)T(t) = 0 \Rightarrow T(t) = A_3 \cos \sqrt{c^2(b_1^2 + b_2^2)}t + B_3 \sin \sqrt{c^2(b_1^2 + b_2^2)}t$$

$$\text{Apa } u(x,y,t) = (A_1 \cos b_1 x + B_1 \sin b_1 x)(A_2 \cos b_2 y + B_2 \sin b_2 y)(A_3 \cos \sqrt{c^2(b_1^2 + b_2^2)}t + B_3 \sin \sqrt{c^2(b_1^2 + b_2^2)}t)$$

$$\sqrt{c^2(b_1^2 + b_2^2)} = \sqrt{c^2(b_1^2 + b_2^2)}$$

$$\text{f.v. } x=0 \rightarrow u(0,y,t) = 0 \Rightarrow A_1 = 0$$

$$\text{f.v. } y=0 \rightarrow u(x,0,t) = 0 \Rightarrow A_2 = 0$$

$$\text{dus } u(x,y,t) = B_1 \sin b_1 x \cdot B_2 \sin b_2 y (A_3 \cos \sqrt{c^2(b_1^2 + b_2^2)}t + B_3 \sin \sqrt{c^2(b_1^2 + b_2^2)}t) \quad (2)$$

$$\text{f.v. } x=L \rightarrow u(L,y,t) = 0 \Rightarrow B_2 \sin b_2 y = 0 \Rightarrow b_2 y = n\pi, \quad n=1,2,\dots$$

$$\text{f.v. } y=2 \rightarrow u(x,2,t) = 0 \Rightarrow B_2 \sin b_2 \cdot 2 = 0 \Rightarrow b_2 \cdot 2 = m\pi \Rightarrow B_2 = \frac{m\pi}{2}, \quad m=1,2,\dots$$

opg v(2) voorwaarde:

$$u(x,y,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} B_n \sin(n\pi x) B_m \sin\left(\frac{m\pi}{2} y\right) (A_{nm} \cos \sqrt{c^2(b_1^2 + b_2^2)} t + B'_{nm} \sin \sqrt{c^2(b_1^2 + b_2^2)} t) \quad (3)$$

$$\text{Oef v(3) voorwaarde: f.v. } t=0 \rightarrow u(x,y,0) = 3 \sin 2\pi x \sin 3\pi y$$

$$\text{gegeven dat } t=0 \rightarrow u(x,y,0) = 3 \sin 2\pi x \sin 3\pi y \text{ dan: } B_{nm} = 3, \quad 2n = n\pi, \quad 3m = \frac{m\pi}{2}, \quad A'_{26} \cos(0) = 1$$

$$B_{26} = 3 \quad n=2 \quad m=6 \quad A'_{26} = 1$$

$$\text{opg v(3) voorwaarde: } u(x,y,t) = 3 \sin 2\pi x \sin 3\pi y (\cos \sqrt{c^2(b_1^2 + b_2^2)} t + B'_{26} \sin \sqrt{c^2(b_1^2 + b_2^2)} t)$$

$$\text{van } 3 \times 0 \text{ is } u_t(x,y,t) = 3 \sin 2\pi x \sin 3\pi y (-B'_{26} \sin \sqrt{c^2(b_1^2 + b_2^2)} t + B''_{26} \cos \sqrt{c^2(b_1^2 + b_2^2)} t)$$

$$\text{f.v. } t=0 \rightarrow u_t(x,y,0) = 0 \Rightarrow -B'_{26} \sin(0) + B''_{26} \cos(0) = 0 \Rightarrow B''_{26} = 0 \text{ dus}$$

$$\boxed{u(x,y,t) = 3 \sin 2\pi x \sin 3\pi y \cos \sqrt{c^2(b_1^2 + b_2^2)} t}$$

1816120 Opgave 2 (Korteweg)

Na dat dit zo mogelijk een oplossing van het probleem zijn.

$$(1) \Delta u(x,y,t) = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} u(x,y,t), \quad 0 < x < 3, \quad 0 < y < 2, \quad t > 0$$

$$u(0,y,t) = u(3,y,t) = u(x,0,t) = u(x,2,t) = 0, \quad 0 < x < 3, \quad 0 < y < 2, \quad t > 0$$

$$u(x,y,0) = \sin nx \sin 4ny, \quad 0 < x < 3, \quad 0 < y < 2$$

$$u_t(x,y,0) = 0, \quad 0 < x < 3, \quad 0 < y < 2$$

$$(1) \Rightarrow u_{xx} + u_{yy} - \frac{1}{c^2} u_{ttt} = 0$$

Onderwijs daten dat $u(x,y,t) = X(x)Y(y)T(t)$ van de vorm is voor (1)

$$X''(x)Y(y)T(t) + X(x)Y''(y)T(t) - \frac{1}{c^2} X(x)Y(y)T''(t) = 0 \Rightarrow$$

drukken op $X(x)Y(y)T(t)$

$$\Rightarrow \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} - \frac{1}{c^2} \frac{T''(t)}{T(t)} = 0 \quad \text{van } \frac{X''(x)}{X(x)} = k_1 \quad \frac{Y''(y)}{Y(y)} = k_2, \quad -\frac{1}{c^2} \frac{T''(t)}{T(t)} = k_3$$

\downarrow
 $k_1 + k_2 + k_3 = 0$

$$\Rightarrow X''(x) - k_1 X(x) = 0 \quad \text{van enige oplossing door we nos} \times \text{drukken } k = b^2 < 0$$

$$\text{drukken } X''(x) + b^2 X(x) = 0 \Rightarrow X(x) = A_1 \cos b_1 x + B_1 \sin b_1 x$$

$$\Rightarrow \text{Opgave } Y(y) = A_2 \cos b_2 y + B_2 \sin b_2 y$$

$$\Rightarrow T''(t) + c^2 k_3 T(t) = 0 \Rightarrow T''(t) + c^2 (b_1^2 + b_2^2) T(t) = 0 \Rightarrow$$

$$\Rightarrow T(t) = A_3 \cos \sqrt{c^2(b_1^2 + b_2^2)} t + B_3 \sin \sqrt{c^2(b_1^2 + b_2^2)} t$$

$$\text{drukken } u(x,y,t) = (A_1 \cos b_1 x + B_1 \sin b_1 x)(A_2 \cos b_2 y + B_2 \sin b_2 y)(A_3 \cos t + B_3 \sin t)$$

$$\Rightarrow \sqrt{c^2(b_1^2 + b_2^2)} = \sqrt{c^2(b_1^2 + b_2^2)}$$

$$\text{v.a. } x=0 \Rightarrow A_1 = 0, \quad \text{v.a. } y=0 \Rightarrow A_2 = 0 \quad \text{drukken } u(x,y,t) \text{ voor:}$$

$$u(x,y,t) = B_1 \sin b_1 x \cdot B_2 \sin b_2 y \cdot (A_3 \cos t + B_3 \sin t)$$

$$\text{v.a. } x=3 \Rightarrow u(3,y,t) = 0 \Rightarrow \sin b_1 \cdot 3 = 0 \Rightarrow b_1 \cdot 3 = n\pi \Rightarrow b_1 = \frac{n\pi}{3} \quad n=1,2,\dots$$

$$\text{v.a. } y=2 \Rightarrow u(x,2,t) = 0 \Rightarrow \sin b_2 \cdot 2 = 0 \Rightarrow b_2 = \frac{m\pi}{2} \quad m=1,2,\dots$$

$$\text{Drukken } u(x,y,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} B_n \sin \frac{n\pi}{3} x \cdot B_m \sin \frac{m\pi}{2} y \cdot (A_{nm} \cos t + B'_{nm} \sin t)$$

$$\text{drukken } B_n B_m = B_{nm}$$

$$\text{v.a. } t=0 \Rightarrow u(x,y,0) = \sin nx \sin 4ny \Rightarrow B_{nm} = 1, \quad \frac{n\pi}{3} = n, \quad \frac{m\pi}{2} = 4n, \quad A_{nm} \cos(0) + B'_{nm} \sin(0) \\ \Rightarrow B_{38} = 1, \quad n=3, \quad m=8, \quad A_{38} = 1$$

$$\text{drukken } u(x,y,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sin(nx) \sin(4ny) (\cos t + B'_{38} \sin t)$$

$$\text{van } u_t(x,y,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sin(nx) \sin(4ny) (-4n \sin t + B'_{38} \cos t)$$

$$\text{v.a. } t=0 \Rightarrow u_t(x,y,0) = 0 \Rightarrow -4n \sin(0) + B'_{38} \cos(0) = 0 \Rightarrow B'_{38} = 0$$

$$\text{drukken: } u(x,y,t) = \sin(nx) \sin(4ny) \cos t$$

09/10 Θεμα 5ο (Εξ-διπλωματικός)

$$(1) U_t(x,t) = U_{xx}(x,t) - \cos x \quad 0 < x < 2\pi, t > 0$$

$$U(0,t) = 1 \quad U_x(2\pi,t) = 0 \quad t > 0$$

$$U(x,0) = 3 - \cos x, \quad 0 < x < 2\pi$$

• Θεματική προσέγγιση $U(x,t) = V(x,t) + S(x)$

οντώς έχω:

$$U_t(x,t) = V_t(x,t) \quad \text{και} \quad U_{xx}(x,t) = V_{xx}(x,t) + S''(x)$$

Αρχικό σημείο (1) και έχω

$$V_t(x,t) = V_{xx}(x,t) + S''(x) - \cos x$$

$$\begin{aligned} \text{• Ανατίθεται } S''(x) - \cos x = 0 \Rightarrow S''(x) = \cos x \Rightarrow S(x) = \sin x + C_1 \Rightarrow \\ \Rightarrow S(x) = -\cos x + C_1 x + C_2 \end{aligned}$$

Η $S(x)$ να προστασίει την $V(x,t)$ στην γενού, έχω

$$S(0) = 1 \quad \text{και} \quad S'(2\pi) = 0$$

$$\Rightarrow C_2 = 2 \quad \text{και} \quad C_1 = 0 \quad \text{έπειο} \quad S(x) = -\cos x + 2$$

• Άλλη προσέγγιση: $V(x,t) = U(x,t) - S(x)$ και αυτής γίνεται:

$$V(0,t) = U(0,t) - S(0) = 1 - 1 = 0 \Rightarrow V(0,t) = 0$$

$$V_x(2\pi,t) = U_x(2\pi,t) - S'(2\pi) = 0 \Rightarrow V_x(2\pi,t) = 0$$

$$V(x,0) = U(x,0) - S(x) = 3 - \cos x + \cos x - 2 \Rightarrow V(x,0) = 1$$

$$\left. \begin{array}{l} \text{Απεικονίζεται ριθμός} \\ (2) \quad V_t(x,t) = V_{xx}(x,t), \quad V(0,t) = 0, \quad V_x(2\pi,t) = 0, \quad V(x,0) = 1 \end{array} \right\}$$

$$V(x,t) = X(x)T(t) \xrightarrow{(2)} X(x)T'(t) = X''(x)T(t) \Rightarrow \frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = C$$

$$\Rightarrow X''(x) - CX(x) = 0 \quad (3) \quad \text{και} \quad T'(t) - CT(t) = 0 \quad (4)$$

Για την (3) είναι δύσκολο να βρεθεί λύση ως προς x , δικαιούται $C = -k^2 < 0$ έπειο

$$X''(x) + k^2 X(x) = 0 \Rightarrow X(x) = A \cos kx + B \sin kx$$

Για την (4): $T'(t) = CT(t) \Rightarrow T(t) = C' e^{-kt}$

$$\text{Έπειο} \quad V(x,t) = (A \cos kx + B \sin kx) C' e^{-kt} \quad \text{και} \quad A' = A \cdot C', \quad B' = B \cdot C'$$

$$V(x,t) = (A' \cos kx + B' \sin kx) e^{-kt}$$

Για $x=0 \Rightarrow V(0,t) = 0 \Rightarrow A' = 0 \quad \text{έπειο} \quad V(x,t) = B' \sin kx e^{-kt}$

$$\text{Για} \quad x=2\pi \Rightarrow V(2\pi,t) = 0 \Rightarrow \cos 2\pi k = 0 \Rightarrow 2\pi k = n\pi - \frac{\pi}{2} \Rightarrow k_n = \frac{2n-1}{4}, \quad n \in \mathbb{N}_+$$

$$\text{έπειο} \quad V(x,t) = \sum_{n=1}^{\infty} B'_n \sin k_n x e^{-k_n t}$$

$$\text{Για} \quad t=0 \Rightarrow V(x,0) = 1 \Rightarrow \sum_{n=1}^{\infty} B'_n \sin k_n x = 1 \Rightarrow$$

$$\Rightarrow \sum_{n=1}^{2n} B'_n \sin k_n x \cdot \sin k_n x dx = \int_0^{2\pi} \sin k_n x dx \Rightarrow (\text{για } n=m)$$

$$\Rightarrow B'_n \int_0^{2\pi} \sin^2 k_n x dx = \int_0^{2\pi} \sin k_n x dx = B'_n \int_0^{2\pi} (1 - \cos^2 k_n x) dx = -\frac{1}{k_n} \cos k_n x \Big|_0^{2\pi}$$

$$\Rightarrow B'_n \left(\int_0^{2\pi} dx - \int_0^{2\pi} \frac{\cos 2k_n x}{2} dx - \int_0^{2\pi} \frac{1}{2} dx \right) = -\frac{1}{k_n} \left(\cos \frac{2\pi n}{2} - 1 \right) \Rightarrow$$

$$\Rightarrow B_n \left(n - \frac{1}{2} \int_0^{2\pi} \cos 2knx dx \right) = -\frac{1}{kn} (\cos(n\pi - \frac{\pi}{2}) - 1)$$

06/09 07:40 5^o (eigener Differenz)

$$\text{Integrating } \int_0^{2\pi} \cos 2knx dx:$$

$$\int_0^{2\pi} \cos 2knx dx = \frac{1}{2kn} \sin 2knx \Big|_0^{2\pi} = 0 \text{ ergo } B_n \pi = \frac{1}{kn} \Rightarrow B_n = \frac{1}{\pi kn}$$

$$\text{Also } V(x,t) = \sum_{n=1}^{\infty} \frac{1}{\pi kn} \sin knx e^{-kn^2 t} \text{ von rechts}$$

$$U(x,t) = V(x,t) + S(x) \Rightarrow U(x,t) = \sum_{n=1}^{\infty} \frac{1}{\pi kn} \sin knx e^{-kn^2 t} - \cos x + 2$$

$$U_t(x,t) = U_{xx}(x,t) - 2T, \quad 0 < x < L, \quad t > 0$$

$$U_x(0,t) = L, \quad U(L,t) = L^2, \quad t > 0$$

$$U(x,0) = Tx^2 + Lx, \quad 0 < x < L$$

- Differenzieren nach x ergibt $U(x,t) = V(x,t) + S(x)$ (2)

$$U_t(x,t) = V_t(x,t) \quad \text{und} \quad U_{xx}(x,t) = V_{xx}(x,t) + S''(x)$$

Ausdrücken von (1) nach $S''(x)$

$$V_t(x,t) = V_{xx}(x,t) + S''(x) - 2T.$$

- Ansetzen $S''(x) - 2T = 0 \Rightarrow S''(x) = 2T \Rightarrow S(x) = 2Tx^2 + C_1 \Rightarrow S(x) = Tx^2 + C_1 + C_2$

nach $S(x)$ nahezu zu interessant ist nur in $U(x,t)$ ganz einfacher:

$$\text{f. z. } x=0 \Rightarrow S'(0)=L \Rightarrow C_1=L$$

$$\text{f. z. } x=L \Rightarrow S(L)=L^2 \Rightarrow TL^2 + L^2 + C_2 = L^2 \Rightarrow C_2 = -TL^2$$

$$\Rightarrow S(x) = Tx^2 + Lx - TL^2$$

- Aus (2) folgt $V(x,t) = U(x,t) - Tx^2 - Lx + TL^2$

$$V_x(0,t) = U_x(0,t) \quad \bar{V}_x(0,t) = L - L \Rightarrow V_x(0,t) = 0$$

$$V(L,t) = U(L,t) - S(L) \Rightarrow V(L,t) = L^2 - L^2 \Rightarrow V(L,t) = 0$$

$$V(x,0) = U(x,0) - S(x) \Rightarrow V(x,0) = Tx^2 + Lx - Tx^2 - Lx + TL^2 \Rightarrow V(x,0) = TL^2$$

$$\left. \begin{array}{l} \text{Apa für } V(x,t) \text{ zu wählen:} \\ \left\{ \begin{array}{l} V_t(x,t) = V_{xx}(x,t), V_x(0,t) = 0, V(L,t) = 0, V(x,0) = TL^2 \end{array} \right. \end{array} \right\}$$

Dann aus $V(x,t) = X(x)T(t)$ folgt

$$X''(x)T(t) = X''(x)T'(t) \Rightarrow \frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)} = C \Rightarrow X''(x) - CX(x) = 0 \quad (3)$$

und $T'(t) = CT(t) \quad (4)$

Ordnungsdifferenzialgleichung der Ordnung 2 mit $C = k^2 < 0$:

$$X''(x) + k^2 X(x) = 0 \Rightarrow X(x) = A \cos kx + B \sin kx$$

$$\text{Mit (4) erhalten dann zur } T(t) = C_1 e^{ct} \Rightarrow T(t) = C_1 e^{-k^2 t} \text{ ordne zu:}$$

$$V(x,t) = (A \cos kx + B \sin kx) C_1 e^{-k^2 t} \text{ und } A' = A C_1, B' = B C_1:$$

$$V(x,t) = (A' \cos kx + B' \sin kx) e^{-k^2 t}$$

$$\text{Für } x=0 \Rightarrow V_x(0,t) = (-A' k \sin(k \cdot 0) + B' k \cos(k \cdot 0)) e^{-k^2 t} \text{ ände:}$$

$$\text{geg. } V_x(0,t) = 0 \Rightarrow B' = 0 \text{ ände } V(x,t) = A' \cos kx \cdot e^{-k^2 t}$$

$$\text{Für } x=L \Rightarrow V(L,t) = 0 \Rightarrow A' \cos kL e^{-k^2 t} = 0 \Rightarrow \cos kL = 0 \Rightarrow kL = n\pi - \frac{\pi}{2} \Rightarrow$$

$$\Rightarrow k_n = \frac{2n\pi - \pi}{2L}, n=1, 2, \dots$$

$$\text{ände } V(x,t) = \sum_{n=1}^{\infty} A'_n \cos k_n x \cdot e^{-k_n^2 t}$$

$$\text{Für } t=0 \Rightarrow V(x,0) = TL^2 \Rightarrow \sum_{n=1}^{\infty} A'_n \cos k_n x = TL^2 \Rightarrow$$

$$\Rightarrow \int_0^L \sum_{n=1}^{\infty} A'_n \cos k_n x \cos k_m x dx = \int_0^L TL^2 \cos k_m x dx \Rightarrow (\text{für } n=m)$$

$$\Rightarrow A'_n \int_0^L \cos^2 k_n x dx = \int_0^L TL^2 \cos k_n x dx \Rightarrow A'_n \frac{L}{2} = TL^2 \frac{1}{k_n} \sin k_n x \Big|_0^L$$

$$\Rightarrow \sin\left(\frac{2n\pi - \pi}{2L} L\right) = \sin\left(n\pi - \frac{\pi}{2}\right) = (-1)^n \text{ ände}$$

$$A'_n \frac{L}{2} = TL^2 \frac{1}{k_n} (-1)^n \Rightarrow A'_n = \frac{2TL}{k_n} (-1)^n \text{ ordne}$$

$$V(x,t) = \sum_{n=1}^{\infty} \frac{2TL}{k_n} \cos k_n x e^{-k_n^2 t} \text{ ände zulieb:}$$

$$\boxed{U(x,t) = \sum_{n=1}^{\infty} \frac{2TL}{k_n} \cos k_n x e^{-k_n^2 t} + T x^2 + L x - TL^2}$$

Θερμότητα Θερμίδα:

$$u_t = k(u_{xx} + p(x,t)), \quad 0 < x < a, \quad t > 0$$

$$u(x,0) = f(x), \quad 0 \leq x \leq a \quad (\text{for positive } u_x(0,t) = \underline{0}) \text{ και}$$

$$u(0,t) = l(t), \quad u(a,t) = r(t), \quad t \geq 0 \quad u_x(1,t) = r'(x)$$

04/09/12 Θερμίδα (ΠΕΤ (Poisson)) {Μη διπλής Laplace}

$$\Delta u(p,\varphi) = 2p \cos 2\varphi, \quad 2 \leq p \leq 3, \quad 0 \leq \varphi \leq 2\pi$$

$$\frac{\partial u(2,\varphi)}{\partial p} = A + \sin \varphi, \quad \frac{\partial u(3,\varphi)}{\partial p} = 0, \quad 0 \leq \varphi \leq 2\pi$$

- Η ανάρτηση σε γραμμή $p(x,t)$ αντέπει στη φύση της τον ονοματοθεατή παραγόντα (χερεκαν) θερμότητα και διαφορετικούς (αντίστοιχους) με τον θερμότηταν τον τύπο των πάθων.

Απόχριση της θερμότητας στην πόλη σε σχέση με την πόλη στην πόλη.

$$\text{Άρδευτης καθίσματος για πάθη: } \int_0^3 \int_0^{2\pi} \Delta u dx d\varphi = \int_0^3 \int_0^{2\pi} \nabla u \cdot \nabla dx d\varphi = \int_0^3 \int_0^{2\pi} \frac{\partial u}{\partial n} dS$$

- Η ανάρτηση $f(x)$ παρέχει την επιλογή της θερμότητας της πόλης.

$$\text{Άρδευτης καθίσματος για πάθη: } \int_2^3 \int_0^{2\pi} \Delta u(p,\varphi) \cdot p d\varphi dp = \int_2^3 0 \cdot 3 d\varphi - \int_2^3 (A + \sin \varphi) d\varphi$$

- Οι αναρτήσεις $l(t)$ και $r(t)$ δίνουν τη θερμότητα της πόλης σε κάθε σημείο της πόλης και επίσης την επίπεδη της πόλης.

$$\Rightarrow \int_2^3 \int_0^{2\pi} 2p^2 \cos 2\varphi d\varphi dp = -2A_\varphi - \int_0^{2\pi} 2 \sin \varphi d\varphi \Rightarrow$$

$$\Rightarrow \int_2^3 \int_0^{2\pi} p^2 \sin 2\varphi \Big|_0^{2\pi} dp = -2A_\varphi + 2 \cos \varphi \Big|_0^{2\pi} \Rightarrow 0 = -2A_\varphi \Rightarrow A_\varphi = 0$$

- Γενική εκφραση αυτού στην την εξίσωση της θερμότητας $u(x,t)$ σε διαφορετικές μονάδες για την πόλη.

Από σκοπό της πόλης:

- * Τα $o'(x), r'(x)$ και $\theta'(x)$ είναι ποινικές μονάδες θερμότητας, που εκφράζουν την πόλη και τη διάφορη.

Μηριανής της πόλης σε σχέση με την πόλη.

$$\left\{ \begin{array}{l} \Delta u(p,\varphi) = 2p \cos 2\varphi, \quad 2 \leq p \leq 3, \quad 0 \leq \varphi \leq 2\pi \\ \frac{\partial u(2,\varphi)}{\partial p} = \sin \varphi, \quad \frac{\partial u(3,\varphi)}{\partial p} = 0, \quad 0 \leq \varphi \leq 2\pi \end{array} \right.$$

Καν παρέχει την πόλη σε σχέση με την πόλη:

$$\Delta u(p,\varphi) = U_{pp} + \frac{1}{p} U_p + \frac{1}{p^2} U_{pp}$$

Όμως στον τύπο $U(p, \varphi) = V(p, \varphi) + \phi(p, \varphi)$ η $V(p, \varphi)$ είναι διαν
και $\phi(p, \varphi)$ απόλυτη γενικής
και ανατίθεται $\Delta \phi = 0$ οπότε $\Delta U(p, \varphi) = \Delta V(p, \varphi) = 2p \cos 2\varphi$

$$\text{όπου } \frac{\partial^2 V(p, \varphi)}{\partial p^2} + \frac{1}{p} \frac{\partial V(p, \varphi)}{\partial p} + \frac{1}{p^2} \frac{\partial^2 V(p, \varphi)}{\partial \varphi^2} = 2p \cos 2\varphi$$

η $V(p, \varphi)$ δεν είναι τύπος $A p^3 \cos 2\varphi$ (το p^3 το καταδικάζει
και το p^2 ότι προσαρτάται την τελευταία όπου του A γίνεται και το p του B')

ανεπαργίας: $6Ap \cos 2\varphi + 3Ap \cos 2\varphi - 4Ap \cos 2\varphi = 2p \cos 2\varphi \Rightarrow$
 $\Rightarrow 5A = 2 \Rightarrow A = \frac{2}{5}$

$$\text{όπου } V(p, \varphi) = \frac{2}{5} p^3 \cos 2\varphi$$

Για την ϕ σχολής: $U(p, \varphi) = V(p, \varphi) + \phi(p, \varphi) \Rightarrow \phi(p, \varphi) = U(p, \varphi) - V(p, \varphi)$

$$\frac{\partial \phi(p, \varphi)}{\partial p} = \frac{\partial U(p, \varphi)}{\partial p} - \frac{\partial V(p, \varphi)}{\partial p} \Rightarrow \frac{\partial \phi(p, \varphi)}{\partial p} = \frac{\partial U(p, \varphi)}{\partial p} - \frac{6}{5} p^2 \cos 2\varphi$$

$$\text{για } p=2: \frac{\partial \phi(2, \varphi)}{\partial p} = \sin \varphi - \frac{24}{5} \cos 2\varphi$$

$$\text{για } p=3: \frac{\partial \phi(3, \varphi)}{\partial p} = -\frac{54}{5} \cos 2\varphi$$

Άρα σχολή το πρόβλημα:

$$\left. \begin{array}{l} \Delta \phi(p, \varphi) = 0 \quad 2 < p < 3 \quad 0 \leq \varphi \leq 2\pi \\ \frac{\partial \phi(p, \varphi)}{\partial p} = \sin \varphi - \frac{24}{5} \cos 2\varphi \quad \frac{\partial \phi(3, \varphi)}{\partial p} = -\frac{54}{5} \cos 2\varphi \end{array} \right\} \begin{array}{l} \text{πλέον} \\ \text{προστίθεται} \end{array}$$

Όμως διότι το τύπο $\phi(p, \varphi) = R(p) \Phi(\varphi)$ οπότε
πειραμάτων την σήμερη διάν η ονομαίειν περιοδική.

$$\phi(p, \varphi) = \frac{a_0}{2} + b_0 \ln p + \sum_{n=1}^{\infty} p^n (a_n \cos n\varphi + b_n \sin n\varphi) + \sum_{n=1}^{\infty} p^{-n} (c_n \cos n\varphi + d_n \sin n\varphi)$$

οπότε:

$$\frac{\partial \phi}{\partial p} = \frac{b_0}{p} + \sum_{n=1}^{n-1} p^{n-1} (a_n \cos n\varphi + b_n \sin n\varphi) - \sum_{n=1}^{n-1} p^{-n-1} (c_n \cos n\varphi + d_n \sin n\varphi)$$

• για $p=2$:

$$\frac{b_0}{2} + \sum_{n=1}^{n-1} 2^{n-1} (a_n \cos n\varphi + b_n \sin n\varphi) - \sum_{n=1}^{n-1} 2^{-n-1} (c_n \cos n\varphi + d_n \sin n\varphi) = \sin \varphi - \frac{24}{5} \cos 2\varphi$$

όπως έχουμε

$$b_1 \sin \varphi - \frac{1}{4} d_1 \sin \varphi = \sin \varphi \Rightarrow \boxed{d_1 - \frac{1}{4} d_1 = 1} \quad \text{και}$$

$$2 \cdot 2 a_2 \cos 2\varphi - \frac{1}{8} c_2 \cos 2\varphi = -\frac{24}{5} \cos 2\varphi \Rightarrow \boxed{4a_2 - \frac{1}{8} c_2 = -\frac{24}{5}} \quad (1)$$

$$\text{και } 2^{n-1} a_n = -n p^{n-1} c_n = 0 \text{ για } n \neq 2$$

$$\text{και } 2^{n-1} b_n = -n 2^{n-1} d_n = 0 \text{ για } n \neq 1 \text{ και } \boxed{b_0 = 0} \quad (2)$$

• για $p=3$:

$$\sum_{n=1}^{n-1} 3^{n-1} (a_n \cos n\varphi + b_n \sin n\varphi) - \sum_{n=1}^{n-1} 3^{-n-1} (c_n \cos n\varphi + d_n \sin n\varphi) = -\frac{54}{5} \cos 2\varphi$$

$$\Rightarrow 6a_2 \cos 2\varphi - \frac{2}{27} c_2 \cos 2\varphi = -\frac{54}{5} \cos 2\varphi \Rightarrow \boxed{6a_2 - \frac{2}{27} c_2 = -\frac{54}{5}} \quad (3)$$

$$\text{και } 3^{n-1} b_n = -n 3^{n-1} d_n = 0 \quad \forall n \text{ οποια και } n=1 \text{ οπε}$$

$$\boxed{b_1 = -\frac{1}{9} d_1} \quad (4)$$

$$\begin{cases} \textcircled{1} \Rightarrow b_1 - \frac{1}{4}d_1 = 1 \\ \textcircled{2} \Rightarrow b_1 = -\frac{1}{4}d_1 \end{cases} \quad \left\{ \begin{array}{l} -\frac{1}{4}d_1 - \frac{1}{4}d_1 = 1 \\ \left(-\frac{4}{36} - \frac{9}{36} \right)d_1 = 1 \end{array} \right\} \quad d_1 = -\frac{36}{13}$$

09/08 Θέμα 1 (ΠΣΤ (Poisson) σε κολυσί) { μη αρνικό Laplace }

$$\begin{cases} \textcircled{3} \Rightarrow 4a_2 - \frac{1}{8}c_2 = -\frac{24}{5} \\ c_0 - \frac{2}{27}c_2 = -\frac{54}{5} \end{cases} \quad \left\{ \begin{array}{l} a_2 = \frac{1}{32}c_2 - \frac{6}{5} \\ c_2 = -\frac{7776}{245} \end{array} \right\} \quad \left\{ \begin{array}{l} a_2 = \frac{537}{245} \\ c_2 = \frac{7776}{245} \end{array} \right\}$$

$$\text{άπαξ λέξης } \phi(p, \varphi) = \frac{a_0}{2} + p^2 \frac{537}{245} \cos 2\varphi + p \frac{36}{117} \sin 2\varphi - p^2 \frac{7776}{245} \cos 2\varphi - p \frac{36}{13} \sin 2\varphi$$

$$\left\{ \begin{array}{l} \Delta u(p, \varphi) = p \cos \varphi \quad 0 \leq p < 3 \quad 0 \leq \varphi \leq 2\pi \\ \frac{\partial u(p, \varphi)}{\partial p} \Big|_{p=3} = 5x + A \end{array} \right\}$$

άπαξ τελών:

$$u(p, \varphi) = \frac{a_0}{2} - p^2 \frac{537}{245} \cos 2\varphi + p \frac{36}{117} \sin 2\varphi - p^2 \frac{7776}{245} \cos 2\varphi - p \frac{36}{13} \sin 2\varphi + \frac{2}{5} p^3 \cos 2\varphi$$

$$\text{Αναδύοντας γεμίζοντας: } \int_0^3 \int_0^{2\pi} \nabla u \cdot \nabla u \, d\varphi \, dp = \int_0^3 \int_0^{2\pi} \nabla u \cdot \nabla u \, d\varphi \, dp = \int_0^3 \frac{\partial u}{\partial r} \, dr \, dp$$

άπαξ τελών της λύσης

$$\int_0^3 \int_0^{2\pi} p(\cos \varphi) \cdot p \cdot d\varphi \, dp = \int_0^{2\pi} 3 (5p \cos \varphi + A) \, d\varphi \Rightarrow$$

$$\Rightarrow 0 = 6\pi A \Rightarrow \boxed{A=0} \text{ οπότε } \frac{\partial u(p, \varphi)}{\partial p} \Big|_{p=3} = \frac{15}{8} \cos \varphi$$

Θεωρώντας την πρώτη λύση $u(p, \varphi) = V(p, \varphi) + \phi(p, \varphi)$ η οποία είναι
λύση ως την αρχική συνθήκη.

Ανατίθετο $\Delta \phi = 0$ οπότε $\Delta u(p, \varphi) = \Delta V(p, \varphi) = p \cos \varphi \Rightarrow$

$$\Rightarrow \frac{\partial^2 V(p, \varphi)}{\partial p^2} + \frac{1}{p} \frac{\partial V(p, \varphi)}{\partial p} + \frac{1}{p^2} \frac{\partial^2 V(p, \varphi)}{\partial \varphi^2} = p \cos \varphi.$$

Με $V(p, \varphi) = A p^3 \cos \varphi$ οπότε

$$6A + p \cos \varphi + 3Ap \cos \varphi - Ap \cos \varphi = p \cos \varphi \Rightarrow 6A + 3A - A = 1 \Rightarrow A = \frac{1}{8}$$

$$\text{οπότε } V(p, \varphi) = \frac{1}{8} p^3 \cos \varphi$$

$$\phi(p, \varphi) = u(p, \varphi) - v(p, \varphi) \Rightarrow \frac{\partial \phi(p, \varphi)}{\partial p} = \frac{\partial u(p, \varphi)}{\partial p} - \frac{\partial v(p, \varphi)}{\partial p} = \frac{\partial u(p, \varphi)}{\partial p} - \frac{3}{8} p^2 \cos \varphi$$

οπότε $\phi(p, \varphi) =$

$$\frac{\partial \phi(p, \varphi)}{\partial p} \Big|_{p=3} = 15 \cos \varphi - \frac{27}{8} \cos \varphi = \frac{93}{8} \cos \varphi$$

Aper exw zo npoblymer:

$$\left\{ \Delta \phi(p, q) = 0, \frac{\partial \phi(p, q)}{\partial p} \Big|_{p=3} = \frac{93}{8} \cos q \right\} \xrightarrow{\text{reduzieren}}$$

Thetaipw dwn zus proppis $\phi(p, q) = R(p) \Phi(q)$ exw n jsmui dwn sivai:

$$\phi(p, q) = \frac{a_0}{2} + b_0 \ln p + \sum_{n=1}^{\infty} p^n (a_n \cos nq + b_n \sin nq) + \sum_{n=1}^{\infty} p^{-n} (c_n \cos nq + d_n \sin nq)$$

zo p jnopsi ro sivai jndiv aper o1 opos $\frac{1}{p^n} \rightarrow \infty$ onorez Thetaipw

$$c_n = d_n = 0$$

afer exw

$$\phi(p, q) = \frac{a_0}{2} + b_0 \ln p + \sum_{n=1}^{\infty} p^n (a_n \cos nq + b_n \sin nq) \text{ uar jua } p=3:$$

$$\frac{\partial \phi(3, q)}{\partial p} = \frac{93}{8} \cos q \Rightarrow \frac{b_0}{3} + \sum_{n=1}^{\infty} n \cdot 3^{n-1} (a_n \cos nq + b_n \sin nq) = \frac{93}{8} \cos q$$

$$\text{afer } \boxed{b_0 = b_n = 0} \text{ uar jua } n=1: a_1 \cos q = \frac{93}{8} \cos q \text{ afer } \boxed{a_1 = \frac{93}{8}}$$

$$\text{afer } \phi(p, q) = \frac{a_0}{2} + p \frac{93}{8} \cos q \text{ uar sivai } U_{(p, q)} = V_{(p, q)} + \phi(p, q)$$

Exkufe zjwai:

$$U(p, q) = \frac{1}{8} p^3 \cos q + \frac{a_0}{2} + p \frac{93}{8} \cos q$$

O9/10 Thetaipw L² (NET (Poisson) or representation) {jn dngers Lophere}

$$\left\{ \begin{array}{l} U(x_1, x_2) = 4 \quad 0 < x_1 < 2 \quad 0 < x_2 < 2 \\ U(0, x_2) = U(2, x_2) = U(x_1, 0) = 0 \\ U(x_1, 2) = 2 \sin \frac{3\pi}{2} x_1 \end{array} \right\}$$

Thetaipw dwn zus proppis:

$$U(x_1, x_2) = V(x_1, x_2) + W(x_1, x_2) \text{ onorez n sivai:}$$

$$V_{x_1 x_1} + W_{x_1 x_1} + V_{x_2 x_2} + W_{x_2 x_2} = 4 \text{ uar sivai:}$$

$$W_{x_1 x_1} + W_{x_2 x_2} = 4 \text{ onorez } V_{x_1 x_1} + V_{x_2 x_2} = 0$$

$$\left\{ \begin{array}{l} \text{Edw Thetaipw ro sivai: } \\ \text{neu ro sivai: } W_{x_1 x_1} + W_{x_2 x_2} = 4 \text{ uar zis proppis:} \end{array} \right.$$

Thetaipw dwn zus proppis:

$$W(x_1, x_2) = a_1 x_1^2 + a_2 x_2^2 + a_3 x_1 x_2 + a_4 x_1 + a_5 x_2 + a_6$$

$$\forall \sum W_{x_1 x_1} = 2a_1 \text{ uar } W_{x_2 x_2} = 2a_2 \text{ afer}$$

$$2a_1 + 2a_2 = 4 \Rightarrow a_1 + a_2 = 2$$

$$W(0, x_2) = 0 \Rightarrow a_2 x_2^2 + a_5 x_2 + a_6 = 0 \Rightarrow \boxed{a_2 = a_5 = a_6 = 0} \text{ uar } \boxed{a_1 = 2}$$

$$W(2, x_2) = 0 \Rightarrow 4a_1 + a_2 x_2^2 + a_3 \cdot 2 x_2 + a_4 \cdot 2 + a_5 x_2 + a_6 = 0 \Rightarrow$$

$$\Rightarrow a_3 \cdot 2 x_2 + 4 \cdot 2 + 2a_4 = 0 \Rightarrow \boxed{a_3 = 0} \text{ uar } \boxed{a_4 = -4}$$

$$\text{uam zjwai } \boxed{W(x_1, x_2) = 2x_1^2 - 4x_1}$$

$$\text{Onöres } V = V(x_1, x_2) + 2x_1^2 - 4x_1 \Rightarrow V(x_1, x_2) = U(x_1, x_2) - 2x_1^2 + 4x_1$$

$$\text{Für } x_1=0: V(0, x_2) = U(0, x_2) \Rightarrow \boxed{V(0, x_2) = 0}$$

$$\text{Für } x_1=2: V(2, x_2) = U(2, x_2) - 2 \cdot 4 + 4 \cdot 2 \Rightarrow \boxed{V(2, x_2) = 0}$$

$$\text{Für } x_2=0: V(x_1, 0) = U(x_1, 0) - 2x_1^2 + 4x_1 \Rightarrow \boxed{V(x_1, 0) = -2x_1^2 + 4x_1}$$

$$\text{Für } x_2=2: V(x_1, 2) = U(x_1, 2) - 2x_1^2 + 4x_1 \Rightarrow \boxed{V(x_1, 2) = 2 \sin \frac{3\pi x_1}{2} - 2x_1^2 + 4x_1}$$

Onöres Fällen zu nördlichen:

$$\left. \begin{aligned} V_{xx_1} + V_{xx_2} &= 0, V(0, x_2) = 0, V(2, x_2) = 0 \\ V(x_1, 0) &= -2x_1^2 + 4x_1 \\ V(x_1, 2) &= 2 \sin \frac{3\pi x_1}{2} - 2x_1^2 + 4x_1 \end{aligned} \right\} \begin{array}{l} \text{replace } x_2 \text{ with } 2 \\ \text{use } x_1 \in [0, 2] \end{array}$$

Einige Lösungen für $V(x_1, x_2) = X_1(x_1)X_2(x_2)$ Onöres sind

$$X_1''(x_1)X_2(x_2) + X_1(x_1)X_2''(x_2) = 0 \Rightarrow$$

$$\Rightarrow \frac{X_1''(x_1)}{X_1(x_1)} + \frac{X_2''(x_2)}{X_2(x_2)} = 0 \Rightarrow \frac{X_1''(x_1)}{X_1(x_1)} = -\frac{X_2''(x_2)}{X_2(x_2)} = C \text{ da}$$

$$X_1''(x_1) - CX_1(x_1) = 0 \text{ und } X_2''(x_2) + CX_2(x_2) = 0$$

Für x_1 Onöres $X_1(x_1) = A_1 \cos kx_1 + B_1 \sin kx_1$ da $C = -k^2 < 0$

$$X_1''(x_1) + k^2 X_1(x_1) = 0 \Rightarrow$$

$$X_1(x_1) = A_1 \cos kx_1 + B_1 \sin kx_1$$

Für x_2 Onöres:

$$X_2''(x_2) - k^2 X_2(x_2) = 0 \Rightarrow$$

$$X_2(x_2) = C_1 e^{kx_2} + D_1 e^{-kx_2}$$

$$\text{Onöres } V(x_1, x_2) = (A_1 \cos kx_1 + B_1 \sin kx_1)(C_1 e^{kx_2} + D_1 e^{-kx_2})$$

$$\text{Für } x_1=0: V(0, x_2) = 0 \Rightarrow A_1(C_1 e^{kx_2} + D_1 e^{-kx_2}) = 0 \text{ da}$$

$$A_1 = 0 \text{ und } C_1 = D_1 = 0 \text{ (ausgenommen) da } \boxed{A_1 = 0}$$

$$\text{Für } x_1=2: V(2, x_2) = 0 \Rightarrow B_1 \sin 2k = 0 \Rightarrow 2k = n\pi \Rightarrow k = \frac{n\pi}{2}$$

man Onöres $C = C_1 B_1$, $D = D_1 B_1$ ist:

$$V(x_1, x_2) = \sum_{n=0}^{\infty} \sin \frac{n\pi}{2} x_1 (C e^{\frac{n\pi x_2}{2}} + D e^{-\frac{n\pi x_2}{2}})$$

$$\text{Für } x_2=0: V(x_1, 0) = -2x_1^2 + 4x_1 \Rightarrow \sum_{n=0}^{\infty} \sin \frac{n\pi}{2} x_1 (C e^{\frac{n\pi x_2}{2}} + D e^{-\frac{n\pi x_2}{2}}) = -2x_1^2 + 4x_1 \Rightarrow$$

$$\Rightarrow (C+D) \int_0^2 \sin^2 \frac{n\pi}{2} x_1 dx_1 = \int_0^2 (-2x_1^2 + 4x_1) \sin \frac{n\pi}{2} x_1 dx_1 \Rightarrow$$

$$\Rightarrow (C+D) \int_0^2 \frac{1}{2} - \frac{\cos n\pi x_1}{2} dx_1 = \frac{2}{n\pi} \int_0^2 (-2x_1^2 + 4x_1) d(\cos \frac{n\pi}{2} x_1) \Rightarrow$$

$$\Rightarrow (C+D) \left(1 - \int_0^2 \sin \frac{2\pi n}{2} x_1 - 1 \right) = \frac{2}{n\pi} (-2x_1^2 + 4x_1) (\cos \frac{n\pi}{2} x_1) \int_0^2 \frac{2}{n\pi} \cos \frac{n\pi}{2} x_1 d(-2x_1^2 + 4x_1)$$

0

Για πολύρρεας σ. 1.11 Απ. 11 (ΠΣΤ (Poisson) σε υποστρώματα) {μη όμοιωση Laplace}

$$\Rightarrow 2(C+D) = \frac{2}{n\pi} \int_0^2 \cos \frac{n\pi}{2} x_1 (-4x_1 + 4) dx_1 \Rightarrow$$

$$\Rightarrow 2(C+D) = \frac{4}{n^3\pi^2} \int_0^2 -4x_1 + 4 d(\sin \frac{n\pi}{2} x_1) \Rightarrow$$

$$\Rightarrow 2(C+D) = \frac{4}{n^3\pi^2} (-4x_1 + 4) (\sin \frac{n\pi}{2} x_1) \Big|_0^2 - \frac{4}{n^3\pi^2} \int_0^2 \sin \frac{n\pi}{2} x_1 d(-4x_1 + 4) \Rightarrow$$

$$\Rightarrow 2(C+D) = \frac{16}{n^3\pi^2} \sin \frac{n\pi}{2} x_1 dx_1 \Rightarrow 2(C+D) = -\frac{32}{n^3\pi^3} (\cos \frac{n\pi}{2} 2 + \cos \frac{n\pi}{2} 0) \Rightarrow$$

$$\Rightarrow 2(C+D) = -\frac{32}{n^3\pi^3} (-1)^n + \frac{32}{n^3\pi^3} (1)$$

$$\begin{cases} U_{xx} + U_{yy} = -\sin(\frac{\pi x}{3}) \cos(\frac{\pi y}{2}), \quad 0 < x < 3, \quad 0 < y < 2 \\ U(0, y) = U(3, y) = 0 \\ U(x, 0) = U(x, 2) = 0 \end{cases}$$

$$\text{για } x=2: V(x, 2) = 2 \sin \frac{3\pi x_1}{2} - 2x_1^2 + 4x_1 = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} \sin \frac{n\pi}{2} x_1 (Ce^{nx} + De^{-nx}) = 2 \sin \frac{3\pi x_1}{2} - 2x_1^2 + 4x_1 \Rightarrow$$

$$\Rightarrow (Ce^{nx} + De^{-nx}) \int_0^2 \sin \frac{n\pi}{2} x_1 dx_1 = \int_0^2 2 \sin \frac{3\pi x_1}{2} - 2x_1^2 + 4x_1 dx_1 \Rightarrow$$

$$\Rightarrow (Ce^{nx} + De^{-nx}) \int_0^2 \sin^2 \frac{n\pi}{2} x_1 dx_1 = \int_0^2 2 \sin \frac{3\pi x_1}{2} \sin \frac{n\pi}{2} x_1 + \int_0^2 (-2x_1^2 + 4x_1) \sin \frac{n\pi}{2} x_1 dx_1$$

$$\boxed{\begin{aligned} I &= 2 \int_0^2 \sin \frac{3\pi x_1}{2} \sin \frac{n\pi x_1}{2} dx_1 = -\frac{4}{n\pi} \int_0^2 \sin \frac{3\pi x_1}{2} d(\cos \frac{n\pi x_1}{2}) = -\frac{4}{n\pi} \sin \frac{3\pi x_1}{2} \cos \frac{n\pi x_1}{2} \Big|_0^2 + \\ &+ \frac{6}{n} \int_0^2 \cos \frac{3\pi x_1}{2} \cos \frac{n\pi x_1}{2} dx_1 = \frac{12}{n^2\pi} \int_0^2 \cos \frac{3\pi x_1}{2} d(\sin \frac{n\pi x_1}{2}) = \\ &= \frac{12}{n^2\pi} \cos \frac{3\pi x_1}{2} \sin \frac{n\pi x_1}{2} \Big|_0^2 - \frac{18}{n^2} I \Rightarrow I = -\frac{18}{n^2} I \Rightarrow I = 0 \end{aligned}}$$

0

αρα

$$2(Ce^{nx} + De^{-nx}) = -\frac{32}{n^3\pi^3} (-1)^n + \frac{32}{n^3\pi^3} (2)$$

$$(1)=(2) \Rightarrow 2(Ce^{nx} + De^{nx}) = 2(C+D) \Rightarrow Ce^{nx} - C = D - De^{-nx} \Rightarrow C = \frac{D(1-e^{-nx})}{(e^{nx}-1)} \quad (3)$$

$$(3) \rightsquigarrow 2(D \left[\frac{1-e^{-nx}}{e^{nx}-1} \right] + D) = -\frac{32}{n^3\pi^3} (-1)^n + \frac{32}{n^3\pi^3} \Rightarrow D = \frac{-\frac{32}{n^3\pi^3} (-1)^n + \frac{32}{n^3\pi^3}}{2 \left[\frac{1-e^{-nx}}{e^{nx}-1} + 1 \right]}$$

$$(3) \Rightarrow C = \frac{-\frac{32}{n^3\pi^3} (-1)^n + \frac{32}{n^3\pi^3}}{2 \left[\frac{1-e^{-nx}}{e^{nx}-1} + 1 \right]} \frac{1-e^{-nx}}{e^{nx}-1} \quad \text{αρα τώρα:}$$

$$(U(x_1, x_2)) = 2x_1^2 - 4x_1 + \sum_{n=0}^{\infty} \sin \frac{n\pi}{2} \left(\left\{ \frac{-\frac{32}{n^3\pi^3} (-1)^n + \frac{32}{n^3\pi^3}}{2 \left[\frac{1-e^{-nx}}{e^{nx}-1} + 1 \right]} \frac{1-e^{-nx}}{e^{nx}-1} \right\} e^{\frac{nx}{2} x_2} + \left\{ \frac{-\frac{32}{n^3\pi^3} (-1)^n + \frac{32}{n^3\pi^3}}{2 \left[\frac{1-e^{-nx}}{e^{nx}-1} + 1 \right]} \right\} e^{-\frac{nx}{2} x_2} \right)$$

για διαφάνεια τη γέφυρα $U(x, y) = V(x, y) + W(x, y)$ αντε νέα γέφυρα:

$$V_{xx} + V_{yy} + W_{xx} + W_{yy} = -\sin(\frac{\pi x}{3}) \cos(\frac{\pi y}{2}) \text{ και αναλογικά:}$$

$$W_{xx} + W_{yy} = -\sin(\frac{\pi x}{3}) \cos(\frac{\pi y}{2}) \text{ αντε } V_{xx} + V_{yy} = 0$$

Οι διαφάνειες αναλογίας γεφύρας $W(x, y)$ και νέα γέφυρα για

$$W_{xx} + W_{yy} = -\sin(\frac{\pi x}{3}) \cos(\frac{\pi y}{2}) \text{ και αναλογίας της διαφάνειας}$$

και ενεργειακής γεφύρας Sint, cost αντε φύγει για γέφυρα, γεφύρα βαθύτερη στην $W(x, y)$ σίγουρα για γέφυρα:

$$W(x, y) = A \sin(\frac{\pi x}{3}) \cos(\frac{\pi y}{2}) \Leftrightarrow$$

$$W_{xx} = A \frac{\pi^2}{9} \sin(\frac{\pi x}{3}) \cos(\frac{\pi y}{2}), \quad W_{yy} = -A \frac{\pi^2}{4} \sin(\frac{\pi x}{3}) \cos(\frac{\pi y}{2}) \text{ αντε διαφάνεια:}$$

$$-A \frac{\pi^2}{9} \sin(\frac{\pi x}{3}) \cos(\frac{\pi y}{2}) - A \frac{\pi^2}{4} \sin(\frac{\pi x}{3}) \cos(\frac{\pi y}{2}) = \sin(\frac{\pi x}{3}) \cos(\frac{\pi y}{2}) \Rightarrow A = \frac{36}{13\pi^2}$$

$$\text{Αρα } W(x, y) = \frac{36}{13\pi^2} \sin(\frac{\pi x}{3}) \cos(\frac{\pi y}{2}) \text{ και } V(x, y) = U(x, y) - \frac{36}{13\pi^2} \sin(\frac{\pi x}{3}) \cos(\frac{\pi y}{2})$$

$$\text{για } x=0: V(0, y) = U(0, y) - 0 \Rightarrow V(0, y) = 0$$

$$\text{για } y=3: V(3, y) = U(3, y) - 0 \Rightarrow V(3, y) = 0$$

$$\text{για } y=0: V(x, 0) = U(x, 0) - \frac{36}{13\pi^2} \sin(\frac{\pi x}{3}) \Rightarrow V(x, 0) = -\frac{36}{13\pi^2} \sin(\frac{\pi x}{3})$$

$$\text{για } y=2: V(x, 2) = U(x, 2) + \frac{36}{13\pi^2} \sin(\frac{\pi x}{3}) \Rightarrow V(x, 2) = \frac{36}{13\pi^2} \sin(\frac{\pi x}{3})$$

Apa svolgjse za rješitev:

$$\left. \begin{aligned} V_{xx} + V_{yy} &= 0, \quad V_{(0,y)} = V_{(3,y)} = 0 \\ V(x,0) &= -\frac{36}{13n^2} \sin\left(\frac{nx}{3}\right), \quad V(x,2) = \frac{36}{13n^2} \sin\left(\frac{nx}{3}\right) \end{aligned} \right\} \text{za različne } n$$

Dsupodjse dogn zns) rješenje $V(x,y) = X(x)Y(y)$ enačte skupje

$$X''(x)Y(y) + X(x)Y''(y) = 0 \Rightarrow \frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = C \text{ van Ostdj.}$$

razredenje) Wgn zor x enačte $C = -k^2$ oper

$$X''(x) + k^2 X(x) = 0 \Rightarrow X(x) = A \cos kx + B \sin kx \text{ van}$$

$$Y''(y) - k^2 Y(y) = 0 \Rightarrow Y(y) = C e^{ky} + D e^{-ky} \text{ enačte:}$$

$$V(x,y) = (A \cos kx + B \sin kx)(C e^{ky} + D e^{-ky})$$

$$\text{pri } x=0: V(0,y) = 0 \Rightarrow A(C e^{ky} + D e^{-ky}) = 0 \Rightarrow A=0 \text{ (zad C=D=0 rješenje)}$$

$$\text{pri } x=3: V(3,y) = 0 \Rightarrow B \sin 3k = 0 \Rightarrow 3k = n_2 \pi \Rightarrow k = \frac{n_2 \pi}{3} \text{ van jde, } C=C' B'$$

$$\text{razredje } V(x,y) = \sum_{n=0}^{+\infty} \sin \frac{n \pi}{3} x (C e^{\frac{n \pi}{3} y} + D e^{-\frac{n \pi}{3} y}) \quad D=D' B'$$

$$\text{pri } y=0: V(x,0) = -\frac{36}{13n^2} \sin\left(\frac{nx}{3}\right) \Rightarrow (C+D) \sum_{n=0}^{+\infty} \sin \frac{n \pi}{3} x = -\frac{36}{13n^2} \sin\left(\frac{nx}{3}\right) \quad \text{pri } 3 \text{ signum obnovit opam}$$

$$\text{razredje: } n=1 \text{ van } C_1 + D_1 = -\frac{36}{13n^2} \quad (1)$$

$$\text{pri } y=2: V(x,2) = \frac{36}{13n^2} \sin\left(\frac{nx}{3}\right) \Rightarrow (C e^{\frac{2n \pi}{3}} + D e^{-\frac{2n \pi}{3}}) \sum_{n=0}^{+\infty} \sin \frac{n \pi}{3} x = \frac{36}{13n^2} \sin\left(\frac{nx}{3}\right)$$

$$\text{van jde slike obnovit opam: } n=1 \text{ van } C e^{\frac{2n \pi}{3}} + D_1 e^{-\frac{2n \pi}{3}} = \frac{36}{13n^2} \Rightarrow C_1 = \frac{36}{13n^2} e^{\frac{-2n \pi}{3}} - D_1 e^{\frac{-2n \pi}{3}} \quad (2)$$

$$(2) \text{ in (1)} \Rightarrow D_1 (1 - e^{-\frac{4n \pi}{3}}) + \frac{36}{13n^2} e^{-\frac{2n \pi}{3}} = -\frac{36}{13n^2} \Rightarrow D_1 = -\frac{36}{13n^2} \frac{(1 + e^{\frac{2n \pi}{3}})}{(1 - e^{-\frac{4n \pi}{3}})} \text{ van } C_1 = -\frac{36}{13n^2} + \frac{36}{13n^2} \frac{(1 + e^{\frac{2n \pi}{3}})}{(1 - e^{-\frac{4n \pi}{3}})}$$

$$\text{oper } V(x,y) = \frac{36}{13n^2} \sin \frac{nx}{3} \cos \frac{ny}{2} + \sin \frac{nx}{3} \left[-\frac{36}{13n^2} + \frac{36}{13n^2} \frac{(1 + e^{\frac{2n \pi}{3}})}{(1 - e^{-\frac{4n \pi}{3}})} e^{\frac{ny}{2}} - \frac{36}{13n^2} \frac{(1 + e^{\frac{2n \pi}{3}})}{(1 - e^{-\frac{4n \pi}{3}})} e^{-\frac{ny}{2}} \right]$$

09/12 Metoda konvergencije Fourier

$$(U_{xx}(x,y) + U_{yy}(x,y)) = 0, \quad -\infty < x < \infty, \quad y > 0$$

$$U(x,0) = f(x), \quad -\infty < x < \infty$$

$U, U_x \rightarrow 0, |x| \rightarrow \infty, \quad y > 0, \quad \text{upravnim} \quad \text{zor} \quad y \rightarrow \infty$

$$\text{Odg} \quad U(x,y) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{y}{y^2 + (x-t)^2} f(t) dt \quad \text{van divozrati:}$$

$$\textcircled{1} \quad F\{U(x,y)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} U(x,t) e^{ist} dt = \hat{U}(s,y)$$

$$\textcircled{2} \quad F^{-1}\{\hat{U}(s,y)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{U}(s,t) e^{-ist} dt = U(s,y)$$

$$\textcircled{3} \quad F\{U_{xx}(x,y)\} = (-is)^2 \hat{U}(s,y)$$

$$\textcircled{4} \quad F^{-1}\{e^{-is}y\} = \frac{1}{\sqrt{2\pi}} \frac{2y}{x^2 + y^2}$$

$$\textcircled{5} \quad F^{-1}\{\hat{f}(s) g(s)\} = (\hat{f} * \hat{g})(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) g(x-t) dt = (g * f)(x)$$

$$\textcircled{6} \quad U_{yy} + U_{xx} = 0 \Rightarrow (-is)^2 \hat{U} + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{d^2 \hat{U}}{dy^2} e^{ist} dx = 0 \Rightarrow$$

$$\Rightarrow \frac{d^2}{dy^2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} U e^{ist} dx - s^2 \hat{U} = 0 \Rightarrow \hat{U}_{yy} - s^2 \hat{U} = 0 \Rightarrow \boxed{\hat{U} = A e^{sy} + B e^{-sy}}$$

$$U(s,0) = f(s) \Rightarrow A + B = f(s) \quad \text{van zor } y \rightarrow \infty, \quad \text{upravnim} \quad \boxed{A=0}$$

$$\text{enako: } \boxed{B=f(s)} \quad \text{oper: } \boxed{\hat{U} = f(s) e^{-sy}}$$

$$\hat{U} = f(s) e^{-sy} \Rightarrow \hat{U} = F\{f(s) * F^{-1}\{e^{-sy}\}\} \Rightarrow \hat{U} = F\{f(s) * \frac{1}{\sqrt{2\pi}} \frac{2y}{x^2 + y^2}\} \Rightarrow$$

$$\Rightarrow F^{-1}\{\hat{U}\} = F^{-1}F\{f(s) * \frac{1}{\sqrt{2\pi}} \frac{2y}{x^2 + y^2}\} \Rightarrow U = f(s) * \frac{1}{\sqrt{2\pi}} \frac{2y}{x^2 + y^2} \Rightarrow$$

$$\Rightarrow U = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) \frac{1}{\sqrt{2\pi}} \frac{2y}{(x-t)^2 + y^2} dt \Rightarrow \boxed{U = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{y}{y^2 + (x-t)^2} f(t) dt}$$

06/10 Metodai pagrindas Fourier

$$\left\{ \begin{array}{l} u_{tt} = u_{xx}, \quad 0 < x < \infty, \quad t > 0, \quad u = u(x, t) \\ u_x(0, t) = -1, \quad u(x, t) = 0, \quad x \rightarrow \infty \\ u(x, 0) = u_t(x, 0) = 0, \quad 0 < x < \infty \end{array} \right.$$

θ d₀ $u = (t-x) H(t-x)$ yje H n^o gurepynas Heaviside dirovau:

$$\left\{ \begin{array}{l} \textcircled{1} F\{u(x, t)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty u(x, t) \cos(sx) ds = \hat{u}(s, t) \\ \textcircled{2} F^{-1}\{\hat{u}(s, t)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{u}(s, t) \cos(sx) ds = u(x, t) \\ \textcircled{3} F\{u_{xx}(x, t)\} = -s^2 \hat{u}(s, t) - \sqrt{\frac{2}{\pi}} u_x(0, t) \\ \textcircled{4} F\{(t-x) H(t-x)\} = \sqrt{\frac{2}{\pi}} \frac{1}{s^2} (1 - \cos(st)) \end{array} \right.$$

$$\begin{aligned} u_{tt} = u_{xx} &\Rightarrow \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{d^2 \hat{u}}{dt^2} \cos(st) dt + s^2 \hat{u}(s, t) + \sqrt{\frac{2}{\pi}} u_x(0, t) = 0 \Rightarrow \\ &\Rightarrow \frac{d^2}{dt^2} \left(\sqrt{\frac{2}{\pi}} \int_0^\infty u \cos(st) dt \right) + s^2 \hat{u} - \sqrt{\frac{2}{\pi}} = 0 \Rightarrow \hat{u}_{tt} + s^2 \hat{u} - \sqrt{\frac{2}{\pi}} = 0 \\ \text{daugiau yje sudinius dirov zmv } \frac{-1/\sqrt{\pi}}{s^2} : & \frac{d^2(-1/\sqrt{\pi})}{dt^2} + s^2 \left(\frac{1/\sqrt{\pi}}{s^2} \right) - \sqrt{\frac{2}{\pi}} = 0 \Rightarrow 0 = 0 \\ \text{van n dirov zmv joforois } \hat{u}_{tt} + s^2 \hat{u} = 0 \text{ ope } \hat{u}_t = A \cos(st) + B \sin(st) \end{aligned}$$

norėz išvadoti $\hat{u} = A \cos(st) + B \sin(st) - \frac{\sqrt{\pi}}{s^2}$ van $\hat{u}_t = -sA \sin(st) + sB \cos(st)$

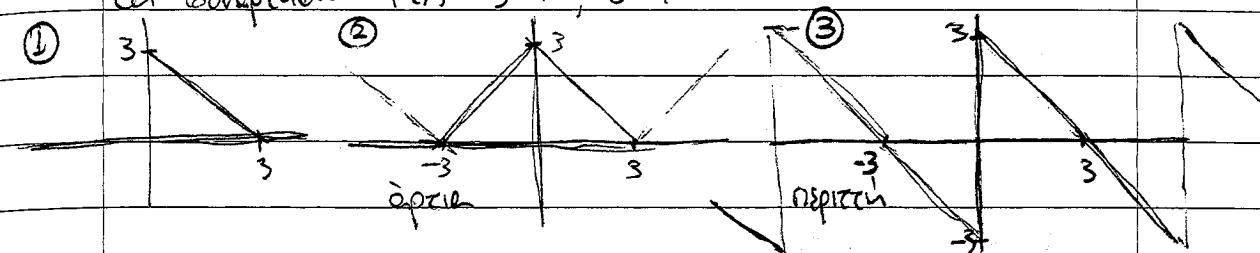
$$\text{jei } t=0: u(s, 0)=0 \Rightarrow A - \frac{\sqrt{\pi}}{s^2} = 0 \Rightarrow A = \frac{\sqrt{\pi}}{s^2} \text{ van } u_t(s, 0)=0 \Rightarrow B=0$$

$$\text{norėz } \boxed{\hat{u} = \frac{\sqrt{\pi}}{s^2} \cos(st) - \frac{\sqrt{\pi}}{s^2}} \Rightarrow \hat{u} = \frac{\sqrt{\pi}}{s^2} (1 - \cos(st)) \Rightarrow F\{u\} = F\{(t-x) H(t-x)\} \Rightarrow$$

$$\Rightarrow F^{-1}\{F\{u\}\} = F^{-1}\{F\{(t-x) H(t-x)\}\} \Rightarrow \boxed{u = (t-x) H(t-x)}$$

09/12 Aritmetyje os grupė Fourier

1) Na opštasi n n^o yjoriumi os grupė Fourier yje
zr gurepynas $f(x) = 3 - x, \quad 0 < x < 3$



žr yje Fourier: $f(x) = \frac{a_0}{2} + \sum (a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L})$, $a_n = \frac{1}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx$
 $a_0 = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx$

$$\begin{aligned} \text{yje zr gurepyni Fourier: } f(x) &= \frac{a_0}{2} + \sum (a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}) \quad \boxed{b_n=0} \\ \text{van } a_n &= \frac{2}{L} \int_0^3 f(x) \cos \frac{n\pi x}{3} dx = \frac{2}{3} \int_0^3 (3 - x) \cos \frac{n\pi x}{3} dx - \frac{2}{3} \int_0^3 x \cos \frac{n\pi x}{3} dx = \\ &= \frac{6}{n\pi} \sin \frac{n\pi x}{3} \Big|_0^3 - \frac{2}{3} \frac{3}{n\pi} \int_0^3 x (\sin \frac{n\pi x}{3})' dx = -\frac{2}{n\pi} \left(x \sin \frac{n\pi x}{3} \Big|_0^3 - \int_0^3 \sin \frac{n\pi x}{3} dx \right) = \\ &= -\frac{2}{n\pi} \cdot \frac{3}{n\pi} \cos \frac{n\pi x}{3} \Big|_0^3 = -\frac{6}{n^2\pi^2} (-1)^n + \frac{6}{n^2\pi^2} = \boxed{-\frac{6}{n^2\pi^2} [(-1)^n - 1]} = a_n \\ a_0 &= \frac{2}{3} \int_0^3 f(x) dx = \frac{2}{3} (3x - \frac{x^2}{2}) \Big|_0^3 = \frac{2}{3} (9 - \frac{9}{2}) = 6 - 3 = \boxed{a_0 = 3} \\ \text{dpa n n^o yjoriumi Fourier vian: } f(x) &= \frac{3}{2} + \sum -\frac{6}{n^2\pi^2} [(-1)^n - 1] \cos \frac{n\pi x}{3} \end{aligned}$$

$$\begin{aligned} \text{yje zr n^o yjoriumi Fourier: } a_0 &= a_n = 0 \text{ van } b_n = \frac{2}{L} \int_0^3 f(x) \sin \frac{n\pi x}{L} dx = \\ &= \frac{2}{3} \int_0^3 3 \sin \frac{n\pi x}{3} dx - \frac{2}{3} \int_0^3 x \sin \frac{n\pi x}{3} dx = -\frac{6}{n\pi} \cos \frac{n\pi x}{3} \Big|_0^3 + \frac{2}{n\pi} \int_0^3 x (\cos \frac{n\pi x}{3})' dx = \\ &= -\frac{6}{n\pi} (-1)^n + \frac{6}{n\pi} + \frac{2}{n\pi} \left(x \cos \frac{n\pi x}{3} \Big|_0^3 - \int_0^3 \cos \frac{n\pi x}{3} dx \right) = \\ &= -\frac{6}{n\pi} (-1)^n + \frac{6}{n\pi} + \frac{6}{n\pi} (-1)^n - \frac{6}{n^2\pi^2} \sin \frac{n\pi x}{3} \Big|_0^3 \Rightarrow \boxed{b_n = \frac{6}{n\pi}} \\ \text{dpa n n^o yjoriumi Fourier vian: } f(x) &= \sum \frac{6}{n\pi} \sin \frac{n\pi x}{3} \end{aligned}$$

2) Na anderlei in joppien van correctieën van nietoriënse gesp's

jie van anderlei in joppien van correctieën van nietoriënse gesp's, Dan
naiputje in correctie van 3 ou gelyksoos, nou sien so reprentie en in die gev
nspelde in gewys. $\int_{-3}^3 -3 < x < 0, f(x) = -x - 3$ van jie $0 < x < 3, f(x) = -x + 3$

$$f(x) = \frac{a_0}{2} + \sum (a_n \cos \frac{n\pi x}{3} + b_n \sin \frac{n\pi x}{3}) \Rightarrow$$

$$\Rightarrow \int_{-3}^3 f(x) dx = \int_{-3}^3 \frac{a_0}{2} dx + \int_{-3}^3 a_n \cos \frac{n\pi x}{3} dx + \int_{-3}^3 b_n \sin \frac{n\pi x}{3} dx =$$

$$= \frac{a_0}{2} \times \int_{-3}^3 + a_n \frac{3}{n\pi} (\sin \frac{n\pi x}{3}) \Big|_{-3}^3 - b_n \frac{3}{n\pi} (\cos \frac{n\pi x}{3}) \Big|_{-3}^3 = \frac{3}{2} a_0 + \frac{3}{n\pi} a_n = 3 a_0$$

$$\text{dple } \int_{-3}^3 f(x) dx = 3 a_0 \Rightarrow a_0 = \frac{1}{3} \int_{-3}^3 f(x) dx = \frac{1}{3} \int_{-3}^0 f(x) dx + \frac{1}{3} \int_0^3 f(x) dx =$$

$$= \frac{1}{3} \int_{-3}^0 -x - 3 dx + \frac{1}{3} \int_0^3 -x + 3 dx = \frac{1}{3} \left(-\frac{x^2}{2} - 3x \right) \Big|_{-3}^0 + \frac{1}{3} \left(-\frac{x^2}{2} + 3x \right) \Big|_0^3 =$$

$$= -\frac{1}{3} \left(-\frac{9}{2} + 9 \right) + \frac{1}{3} \left(-\frac{9}{2} + 9 \right) = 0 \Rightarrow \boxed{a_0 = 0}$$

$$f(x) = \frac{a_0}{2} + \sum (a_n \cos \frac{n\pi x}{3} + b_n \sin \frac{n\pi x}{3}) \Rightarrow \int_{-3}^3 f(x) dx = \int_{-3}^3 \frac{a_0}{2} dx + \int_{-3}^3 a_n \cos \frac{n\pi x}{3} dx +$$

$$+ \int_{-3}^3 b_n \sin \frac{n\pi x}{3} dx \quad \text{van rollendeisw van ce doo } \int_{-3}^3 \sin \frac{n\pi x}{3} dx \Rightarrow$$

$$\Rightarrow \int_{-3}^3 f(x) \sin \frac{n\pi x}{3} dx = \frac{a_0}{2} \int_{-3}^3 \sin \frac{n\pi x}{3} dx + a_n \int_{-3}^3 \cos \frac{n\pi x}{3} \sin \frac{n\pi x}{3} dx + b_n \int_{-3}^3 \sin \frac{n\pi x}{3} \sin \frac{n\pi x}{3} dx$$

$$\text{dple } n=m: \int_{-3}^3 f(x) \sin \frac{n\pi x}{3} dx = b_n \int_{-3}^3 \sin^2 \frac{n\pi x}{3} dx$$

$$\int_{-3}^3 \sin^2 \frac{n\pi x}{3} dx = \int_{-3}^3 1 - \cos \frac{2n\pi x}{3} dx = x \Big|_{-3}^3 - \frac{1}{2} \int_{-3}^3 \cos 2 \frac{n\pi x}{3} dx =$$

$$= 3 + 3 - \frac{1}{2} \times \Big|_{-3}^3 + \frac{2}{2} \sin 2 \frac{n\pi x}{3} \Big|_{-3}^3 = 3 \text{ dple}$$

$$\text{dple } \int_{-3}^3 f(x) \sin \frac{n\pi x}{3} dx = 3 b_n \Rightarrow \boxed{b_n = \frac{1}{3} \int_{-3}^3 f(x) \sin \frac{n\pi x}{3} dx}$$

van jie zo om te klap:

$$f(x) = \frac{a_0}{2} + \sum (a_n \cos \frac{n\pi x}{3} + b_n \sin \frac{n\pi x}{3}) \Rightarrow \int_{-3}^3 f(x) dx = \int_{-3}^3 \frac{a_0}{2} dx + \int_{-3}^3 a_n \cos \frac{n\pi x}{3} dx + \int_{-3}^3 b_n \sin \frac{n\pi x}{3} dx$$

$$\text{van rollendeisw van ce doo } \int_{-3}^3 \cos \frac{n\pi x}{3} dx$$

$$\text{dple } \int_{-3}^3 f(x) \cos \frac{n\pi x}{3} dx = \int_{-3}^3 \frac{a_0}{2} \cos \frac{n\pi x}{3} dx + \int_{-3}^3 a_n \cos \frac{n\pi x}{3} \cos \frac{n\pi x}{3} dx + \int_{-3}^3 b_n \sin \frac{n\pi x}{3} \cos \frac{n\pi x}{3} dx$$

$$\text{van jie } n=m$$

$$\Rightarrow \int_{-3}^3 f(x) \cos \frac{n\pi x}{3} dx = a_n \int_{-3}^3 \cos^2 \frac{n\pi x}{3} dx$$

$$\int_{-3}^3 f(x) \cos \frac{n\pi x}{3} dx = \int_{-3}^0 (-x - 3) \cos \frac{n\pi x}{3} dx + \int_0^3 (-x + 3) \cos \frac{n\pi x}{3} dx =$$

$$= \int_{-3}^0 -x \cos \frac{n\pi x}{3} dx - \int_{-3}^0 3 \cos \frac{n\pi x}{3} dx - \int_0^3 x \cos \frac{n\pi x}{3} dx + \int_0^3 3 \cos \frac{n\pi x}{3} dx =$$

$$= \frac{3}{n\pi} \int_{-3}^0 (-x \sin \frac{n\pi x}{3}) \Big|_0^0 - \frac{3}{n\pi} \int_{-3}^0 3 \sin \frac{n\pi x}{3} dx + \frac{3}{n\pi} \int_0^3 x \sin \frac{n\pi x}{3} dx + \int_0^3 3 \sin \frac{n\pi x}{3} dx =$$

$$= \frac{3}{n\pi} (-x \sin \frac{n\pi x}{3}) \Big|_{-3}^0 + \frac{3}{n\pi} \int_{-3}^0 \sin \frac{n\pi x}{3} dx + \left(\frac{3}{n\pi} x \sin \frac{n\pi x}{3} \right) \Big|_0^3 - \frac{3}{n\pi} \int_0^3 \sin \frac{n\pi x}{3} dx =$$

$$= -\frac{9}{n\pi^2} \cos \frac{n\pi x}{3} \Big|_{-3}^0 + \frac{9}{n\pi^2} \cos \frac{n\pi x}{3} \Big|_0^3 = -\frac{9}{n\pi^2} - \frac{9}{n\pi^2} (-1)^n + \frac{9}{n\pi^2} + \frac{9}{n\pi^2} (-1)^n = 0$$

$$\text{dple } 0 = a_n \int_{-3}^3 \cos^2 \frac{n\pi x}{3} dx \Rightarrow \boxed{a_n = 0}$$

jie in gewys. De naiputje in correctie van lopie van
6xifotoos ②: $\int_{-3}^3 -3 < x < 0, f(x) = x + 3$ van jie $0 < x < 3, f(x) = 3 - x$.

Xpuljeron in die vlooi:

$$\int_{-n}^n \cos nt \cos mt dt = \begin{cases} 0 & \text{naar } n \neq m \\ \pi & \text{naar } n = m \end{cases}$$

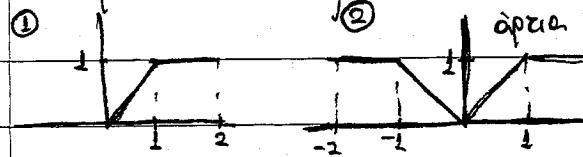
$$\int_{-n}^n \sin nt \sin mt dt = \begin{cases} 0 & \text{naar } n \neq m \\ n & \text{naar } n = m \end{cases}$$

$$\int_{-n}^n \sin nt \cos mt dt = 0 \neq n \neq m$$

10/04 Anwendung der Gleichung Fourier

Die Gleichung der harmonischen Anwendung Fourier zu $f(x) = \begin{cases} x, & 0 \leq x < 1 \\ 1, & 1 \leq x < 2 \end{cases}$

①



②

die Gleichung der harmonischen Anwendung (②)

$$f(x) = \frac{a_0}{2} + \sum (a_n \cos \frac{nx}{2} + b_n \sin \frac{nx}{2})$$

$$\text{zu } a_0 = \frac{2}{2} \int_0^2 f(x) dx = \int_0^1 x dx + \int_1^2 1 dx = \left[\frac{x^2}{2} \right]_0^1 + [x]_1^2 = \frac{3}{2} = a_0$$

$$\begin{aligned} a_n &= \frac{2}{2} \int_0^2 f(x) \cos \frac{nx}{2} dx = \int_0^1 x \cos \frac{nx}{2} dx + \int_1^2 \cos \frac{nx}{2} dx = \\ &= \frac{2}{nn} \int_0^1 x (\sin \frac{nx}{2})' dx + \frac{2}{nn} \sin \frac{nx}{2} \Big|_0^1 = \frac{2}{nn} x \sin \frac{nx}{2} \Big|_0^1 - \frac{2}{nn} \int_0^1 \sin \frac{nx}{2} dx - \frac{2}{nn} \sin \frac{nx}{2} \Big|_0^1 = \\ &= \frac{2}{nn} \sin \frac{nn}{2} - \frac{2}{nn} \sin \frac{nn}{2} + \frac{4}{nn} \cos \frac{nn}{2} \Big|_0^1 \Rightarrow a_n = \frac{4}{nn} (\cos \frac{nn}{2} - 1) \quad \text{zu } b_n = 0 \end{aligned}$$

also
$$f(x) = \frac{3}{4} + \sum \frac{4}{nn} (\cos \frac{nn}{2} - 1) \cos \frac{nx}{2}$$

09/12 Laplace und Gleichung

$$\begin{cases} \Delta u = 0, & 0 \leq r < 2, 0 \leq \theta \leq \pi, 0 \leq \varphi \leq 2\pi \\ u(2, \theta, \varphi) = 2 \cos \theta + 5 \cos^2 \theta \end{cases}$$

$$\text{Lösung Laplace und Gleichung: } \Delta u = U_{rr} + \frac{2}{r} U_r + \frac{1}{r^2} U_{\theta\theta} + \frac{\cos \theta}{r^2 \sin^2 \theta} U_{\varphi\varphi} + \frac{1}{r^2 \sin^2 \theta} U_{\varphi\varphi}$$

$$\text{zu den niedrigsten Legendre: } P_0(x) = 1, P_1(x) = x, P_2(x) = \frac{3x^2 - 1}{2}$$

Anwendung der Gleichung auf die Gleichung $\Delta u = 0$ zu $\cos \theta = x$ und $\cos^2 \theta = x^2$ zu P_0 und P_2 :

$$\begin{aligned} u(2, \theta, \varphi) &= 2 \cos \theta + 5 \cos^2 \theta = 2x + 5x^2 = 2x + 5 \frac{2}{3} \frac{3}{2} x^2 - 5 \frac{2}{3} \left(\frac{1}{2} - \frac{1}{2}\right) = \\ &= 2x + 5 \frac{2}{3} \left(\frac{3x^2 - 1}{2}\right) + \frac{10}{6} = 2P_1 + \frac{10}{3} P_2 + \frac{5}{3} 0 \end{aligned}$$

Rechts zu schreiben

$$\begin{aligned} U_{rr} + \frac{2}{r} U_r + \frac{1}{r^2} U_{\theta\theta} + \frac{\cos \theta}{r^2 \sin^2 \theta} U_{\varphi\varphi} &= 0 \Rightarrow \\ \Rightarrow r^2 \sin^2 \theta U_{rr} + 2r \sin^2 \theta U_r + \sin^2 \theta U_{\theta\theta} + \cos \theta \sin \theta U_{\varphi\varphi} &= 0 \end{aligned}$$

Rechts zu schreiben $U = \Phi(\varphi) \Theta(\theta) R(r)$ anwendbar zu dekomponieren zu U also \dot{x} zu:

$$r^2 \sin^2 \theta \frac{R''(r)}{R(r)} + 2r \sin^2 \theta \frac{R'(r)}{R(r)} + \sin^2 \theta \frac{\Theta''(\theta)}{\Theta(\theta)} + \cos \theta \sin \theta \frac{\Theta'(\theta)}{\Theta(\theta)} = -\frac{\Phi''(\varphi)}{\Phi(\varphi)}$$

\dot{x} zu $-\frac{\Phi''(\varphi)}{\Phi(\varphi)}$ oder \dot{x} zu $\Phi(\varphi)$ zu schreiben zu $\Phi''(\varphi) = 0$

dekomponieren zu $\sin^2 \theta$ zu:

$$r^2 \frac{R''(r)}{R(r)} + 2r \frac{R'(r)}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)} - \frac{\cos \theta}{\sin \theta} \frac{\Theta'(\theta)}{\Theta(\theta)} = C_2 \quad \text{zu } \Theta \text{ zu } C_2 = n(n+1)$$

Aber \dot{x} zu \dot{x} zu \dot{x} zu schreiben:

$$r^2 R''(r) + 2r R'(r) - n(n+1) R = 0 \quad \text{zu } \Theta'' + \frac{\cos \theta}{\sin \theta} \Theta' + n(n+1) \Theta = 0$$

Euler

zu \dot{x} zu schreiben zu Legendre

van curv Euler: $r^2 R''(r) + 2r R'(r) - n(n+1)R = 0$ dan: $R(r) = Ar^n + Br^{-(n+1)}$
 omdat Θ ook $r=0$ van n zijn $r^{-(n+1)} \rightarrow \infty$ omdat $B=0$ opa
 $R(r) = Ar^n$

$$\text{van in } \Theta' + \frac{\cos\theta}{\sin\theta} \Theta' + n(n+1)\Theta = 0 \quad \text{met } x = \cos\theta \Rightarrow dx = -\sin\theta d\theta$$

$$\Rightarrow d\theta = -\frac{1}{\sin\theta} dx = -\frac{1}{\sqrt{1-x^2}} dx \Rightarrow d\theta = -\frac{1}{\sqrt{1-x^2}} dx$$

$$\frac{d\Theta(\theta)}{d\theta} = \frac{d\Theta(x)}{-\frac{1}{\sqrt{1-x^2}} dx} = -\sqrt{1-x^2} \Theta'(x)$$

$$\frac{d^2\Theta(\theta)}{d\theta^2} = \frac{d}{d\theta} (-\sqrt{1-x^2} \Theta'(x)) = -\sqrt{1-x^2} \frac{d}{dx} (-\sqrt{1-x^2} \Theta'(x)) = -\sqrt{1-x^2} \frac{2x}{2\sqrt{1-x^2}} \Theta' + (1-x^2)\Theta'' =$$

$$= -x\Theta' + (1-x^2)\Theta'' \Rightarrow$$

$$\Rightarrow x\Theta' + (1-x^2)\Theta'' + \frac{x}{\sqrt{1-x^2}} (-\sqrt{1-x^2})\Theta' + n(n+1)\Theta = 0 \Rightarrow$$

$$\Rightarrow (1-x^2)\Theta'' - 2x\Theta' + n(n+1)\Theta = 0 \quad (\text{Legendre}) \text{ opa } \Theta = P_n$$

opice n. eigenwaarden: $U = \sum A_n r^n P_n(\cos\theta)$

van Θ opa:

$$U(3,0,4) = 2P_1 + \frac{10}{3}P_2 + \frac{5}{3}P_0 = \sum A_n 2^n P_n \text{ opice:}$$

$$A_0 P_0 = \frac{5}{3} P_0 \Rightarrow \boxed{A_0 = \frac{5}{3}}, A_1 2 \cdot P_1 = 2 \cdot P_1 \Rightarrow \boxed{A_1 = 1}, A_2 \cdot 4 P_2 = \frac{10}{3} P_2 \Rightarrow \boxed{A_2 = \frac{10}{12}}$$

opice tekenie $\boxed{U(r,\theta,4) = \frac{5}{3} + r \cos\theta + \frac{10}{12} r^2 \frac{3 \cos^2\theta - 1}{2}}$

07/12 Laplace os eigenwaarden

$$\left\{ \begin{array}{l} \Delta u = 0, \quad 0 \leq r \leq 5, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \varphi \leq 2\pi \\ u(5, \theta, \varphi) = 5 + 3 \cos\theta + \cos^2\theta \end{array} \right.$$

$$\Delta u = 0 \quad \text{Laplace os eigenwaarden: } U_{rr} + \frac{1}{r} U_{r\theta} + \frac{1}{r^2} U_{\theta\theta} + \frac{\cos\theta}{r^2 \sin^2\theta} U_{\theta\varphi\varphi} + \frac{1}{r^2 \sin^2\theta} U_{\varphi\varphi\varphi} = 0$$

van de radiale vorm Legendre: $P_0(x) = 1, P_1(x) = x, P_2(x) = \frac{3x^2 - 1}{2}$

$$\text{van } x = \cos\theta: \quad U(5, \theta, \varphi) = 5 + 3x + x^2 = 5 + 3x + \frac{2}{3} \frac{3}{2} x^2 - \frac{2}{3} \left(\frac{1}{2} - \frac{1}{2}\right) =$$

$$= 5 + 3x + \frac{2}{3} \left(\frac{3x^2 - 1}{2}\right) + \frac{2}{3} = \frac{32}{6} P_0 + 3 P_1 + \frac{2}{3} P_2$$

Ans van optimaal condum verdeling opa n. eigenwaarden $\Delta u = 0$ opa $U_{\varphi\varphi\varphi} = 0$. Erst rollende vormas $\propto r^2$ n.d.s. pizzi:

$$r^2 U_{rr} + 2r U_{r\theta} + U_{\theta\theta} + \frac{\cos\theta}{\sin\theta} U_{\theta\varphi\varphi} = 0. \quad \text{Opis pizzi dach n. foppas } u = R(r)\Theta(\theta)$$

verdeling opa $\Theta(\theta)$ isoupi:

$$r^2 \frac{R''(r)}{R(r)} + 2r \frac{R'(r)}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)} - \frac{\cos\theta}{\sin\theta} \frac{\Theta'(\theta)}{\Theta(\theta)} = C = n(n+1)$$

H. opa siven curv Euler van bixen jsmui dan curv $R(r) = Ar^n + Br^{-(n+1)}$ alle opis van $r=0$ van zore $r^{-(n+1)} \rightarrow \infty$ opa $B=0$ opa $R(r) = Ar^n$

$$\text{②} \Rightarrow \Theta''(\theta) + \frac{\cos\theta}{\sin\theta} \Theta'(\theta) + n(n+1)\Theta(\theta) = 0 \quad \text{of zw } x = \cos\theta \Rightarrow dx = -\sin\theta d\theta \Rightarrow$$

$$\Rightarrow d\theta = -\frac{1}{\sin\theta} dx = -\frac{1}{\sqrt{1-x^2}} dx, \quad \frac{d\Theta(\theta)}{d\theta} = \frac{d\Theta(x)}{-\frac{1}{\sqrt{1-x^2}} dx} = -\sqrt{1-x^2} \Theta'(x)$$

$$\frac{d^2\Theta(\theta)}{d\theta^2} = -\frac{d}{dx} \left(-\sqrt{1-x^2} \Theta'(x) \right) = -\sqrt{1-x^2} \left(\frac{2x}{2\sqrt{1-x^2}} \Theta'(x) - \sqrt{1-x^2} \Theta''(x) \right) =$$

$$= -x\Theta'(x) + (1-x^2)\Theta''(x) \quad \text{op a}$$

$$-x\theta'(x) + (1-x^2)\theta''(x) + \frac{x}{\sqrt{1-x^2}}(-\sqrt{1-x^2})\theta'(x) + n(n+1)\theta(x) = 0 \Rightarrow$$

$$\Rightarrow (1-x^2)\theta''(x) - 2x\theta'(x) + n(n+1)\theta(x) = 0 \quad (\text{Legendre}) \text{ ope } \theta = P_n$$

ope $U = \sum A_n r^n P_n$ van $r=5$: $\sum A_n 5^n P_n = \frac{32}{6}P_0 + 3P_1 + \frac{3}{2}P_2$

$$A_{00} A_0 P_0 = \frac{32}{6} P_0 \Rightarrow A_0 = \frac{32}{6}$$

$$A_1 5P_1 = 3P_1 \Rightarrow A_1 = \frac{3}{5}$$

$$A_2 25P_2 = \frac{3}{2}P_2 \Rightarrow A_2 = \frac{3}{50}$$

ope $U = \frac{32}{6} + \frac{3}{5}r \cos \theta + \frac{3}{50}r^2 \frac{3 \cos^2 \theta - 1}{2}$

OF/11 Tijfenvo (Kofjevium) vs 2x updrifts gesoluus)

$$(u(p, \varphi, t) = U_{tt}(p, \varphi, t), \quad 0 \leq p < 2, \quad 0 \leq \varphi < 2\pi, \quad t > 0)$$

$$U(2, \varphi, t) = 0, \quad 0 \leq \varphi < 2\pi, \quad t > 0$$

$$u(p, \varphi, 0) = 3J_2(\mu_2, \frac{p}{2}) \sin(2\varphi)$$

$$u(p, \varphi, 0) = 4J_3(\mu_3, \frac{p}{2}) \cos(3\varphi)$$

$c^2 \Delta u = U_{tt}$ Da Δ doops vro van de oplossing $C=1$ (jeet n) waardere

$$c^2 U_{tt} = U_{pp} + \frac{1}{p} U_p + \frac{1}{p^2} U_{ppp} \quad \text{van Drupw } U(p, \varphi) \cdot T(t) = U \Rightarrow$$

$$\Rightarrow \frac{1}{c^2} U_{(p, \varphi)} T''(t) = \frac{\partial^2 U}{\partial p^2} T(t) + \frac{\partial U}{p \partial p} T(t) + \frac{\partial^2 U}{p^2 \partial \varphi^2} T(t) \Rightarrow$$

$$\Rightarrow \frac{1}{c^2} U_{(p, \varphi)} T''(t) = T(t) \Delta u \quad \text{van diaanvrees } \Rightarrow U(p, \varphi) T(t) :$$

$$\frac{1}{c^2} \frac{T''(t)}{T(t)} = \frac{\Delta u}{U(p, \varphi)} = -k^2 \quad \text{ope}$$

$$(T(t)) = C \cos(c k t) + D \sin(c k t)$$

$$\textcircled{1} \quad T''(t) + c^2 k^2 T(t) = 0 \quad \text{van} \quad \textcircled{2} \quad \Delta u + k^2 U(p, \varphi) = 0$$

Hou op dat $T(t) \neq 0$ omdat $U(2, \varphi, t) = 0 \Rightarrow U(2, \varphi) T(t) = 0$ en $T(t) \neq 0$ ope $U(2, \varphi) = 0$

ontruks $\Delta u \neq 0$ ope $U(p, \varphi)$:

$$\left\{ \begin{array}{l} U_{ppp} + \frac{1}{p} U_p + \frac{1}{p^2} U_{ppp} + k^2 U = 0 \\ U(2, \varphi) = 0 \end{array} \right\}$$

$$(U(p, \varphi) = P(p) \Phi(\varphi) \Rightarrow \frac{P''(p)}{P(p)} + \frac{1}{p} \frac{P'(p)}{P(p)} + \frac{1}{p^2} \frac{\Phi''(\varphi)}{\Phi(\varphi)} + k^2 = 0 \Rightarrow$$

$$\Rightarrow p^2 \frac{P''(p)}{P(p)} + p \frac{P'(p)}{P(p)} + k^2 p^2 = -\frac{\Phi''(\varphi)}{\Phi(\varphi)} = n^2 \Rightarrow$$

$$\Rightarrow \textcircled{3} \quad p^2 P''(p) + p P'(p) + P(p) k^2 p^2 - n^2 P(p) = 0 \quad \text{van} \quad \textcircled{4} \quad \Phi''(\varphi) + n^2 \Phi(\varphi) = 0$$

$$\int \Phi(\varphi) = 0$$

$$\textcircled{3}: \rho P''_{(p)} + \rho P'_{(p)} + P_{(p)} (\kappa^2 \rho^2 - n^2) = 0 \quad (\text{Bessel})$$

van $\dot{\rho} \neq 0$ dan:

$$P_{(p)} = A' J_n(\kappa\rho) + B' Y_n(\kappa\rho) \text{ en } \rho=0, Y_n \rightarrow \infty \text{ da}\rho B'=0$$

$$\text{d}\rho P_{(p)} = A' J'_n(\kappa\rho) \text{ van } P_{(0)} = 0 \Rightarrow J'_n(2\kappa) = 0 \text{ da}\rho 2\kappa = \gamma_{nm}$$

$$\text{d}\rho \kappa = \frac{\gamma_{nm}}{2} \text{ van } \infty A' \text{ en } B' \text{ voldoet en da}\rho \text{ j's volgt}$$

oncés:

$$U_{nm}(\rho, \varphi, t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} J_n\left(\frac{\gamma_{nm}}{2}\rho\right) [A_{nm} \sin(n\varphi) + B_{nm} \cos(n\varphi)] \left[C_{nm} \sin\left(\frac{\gamma_{nm}}{2}ct\right) + D_{nm} \cos\left(\frac{\gamma_{nm}}{2}ct\right) \right]$$

da\rho \text{ j's volgt:}

$$U_{nm}(\rho, \varphi, t) = J_{21}\left(\frac{\gamma_{21}}{2}\rho\right) [A_{21} \sin(2\varphi) + B_{21} \cos(2\varphi)] \left[C_{21} \sin\left(\frac{\gamma_{21}}{2}ct\right) + D_{21} \cos\left(\frac{\gamma_{21}}{2}ct\right) \right] + J_{31}\left(\frac{\gamma_{31}}{2}\rho\right) [A_{31} \sin(3\varphi) + B_{31} \cos(3\varphi)] \left[C_{31} \sin\left(\frac{\gamma_{31}}{2}ct\right) + D_{31} \cos\left(\frac{\gamma_{31}}{2}ct\right) \right]$$

$$U(\rho, \varphi, t) = J_{21}\left(\frac{\gamma_{21}}{2}\rho\right) [A_{21} \sin(2\varphi) + B_{21} \cos(2\varphi)] \cdot D_{21} + J_{31}\left(\frac{\gamma_{31}}{2}\rho\right) [A_{31} \sin(3\varphi) + B_{31} \cos(3\varphi)] \cdot D_{31} = 3 J_{21}\left(\frac{\gamma_{21}}{2}\rho\right) \sin(2\varphi) \text{ da}\rho$$

$$D_{21}=0, D_{21} \cdot A_{21}=3 \text{ van } B_{21}=0$$

$$U_t(\rho, \varphi, t) = J_{21}\left(\frac{\gamma_{21}}{2}\rho\right) [A_{21} \sin(2\varphi) + B_{21} \cos(2\varphi)] C_{21} + J_{31}\left(\frac{\gamma_{31}}{2}\rho\right) [A_{31} \sin(3\varphi) + B_{31} \cos(3\varphi)] C_{31} = 4 J_{31}\left(\frac{\gamma_{31}}{2}\rho\right) \cos(3\varphi) \text{ da}\rho$$

$$C_{21}=0, A_{21}=0 \text{ van } B_{31} \cdot C_{31} \cdot \frac{4}{3} = 4$$

van j's volgt en zwr enkel een $C=1$

oncés te volgt:

$$U(\rho, \varphi, t) = 3 J_{21}\left(\frac{\gamma_{21}}{2}\rho\right) \sin(2\varphi) \cos(\gamma_{21}t) + \frac{8}{\gamma_{31}} J_{31}\left(\frac{\gamma_{31}}{2}\rho\right) \cos(3\varphi) \sin(\gamma_{31}t)$$

Gegeven $\Sigma 186$ dat u.G Tijdsverloop (K)periode $\mu 2$ tijdsperiodes gevolgt

$$U_{ttt} = C^2 \Delta u, r \leq 2, t \geq 0$$

$$U(z, \varphi, t) = 0, U(r, \varphi, 0) = 3 J_2\left(\gamma_{21}\frac{r}{2}\right) \sin\gamma_{21}t, U(r, \varphi, 0) = J_4\left(\gamma_{31}\frac{r}{2}\right) \cos\gamma_{31}t$$

$$U_{ttt} = C^2 (U_{rr} + \frac{1}{r} (U_r + \frac{1}{r^2} U_{\varphi\varphi})) \text{ van ondicsoupl} U(r, \varphi, t) = R(r) F(\varphi) T(t)$$

van orthogonaleheid en d.s.:

$$RFT'' = C^2 (F T'' + \frac{1}{r} FT' + \frac{1}{r^2} RT'') \text{ van diaparres} \Rightarrow RTFC^2:$$

$$\frac{1}{C^2} \frac{T''}{T} = \frac{R''}{R} + \frac{R'}{rR} + \frac{F''}{r^2 F} = \kappa \text{ da}\rho \text{ j's volgt:}$$

$$\textcircled{1} \quad \frac{T''}{C^2 T} = \kappa \text{ van } \frac{R''}{R} + \frac{R'}{rR} + \frac{F''}{r^2 F} = \kappa \Rightarrow r^2 \frac{R''}{R} + r \frac{R'}{R} - \kappa r^2 = -\frac{F''}{F} = \mu$$

oncés:

$$\textcircled{2} \quad r^2 \frac{R''}{R} + r \frac{R'}{R} - \kappa r^2 = \mu \text{ van } \textcircled{3} - \frac{F''}{F} = \mu$$

Onderdeel reproducerende deel dus F oncés $\mu = m^2$ da\rho $F'' + m^2 F = 0$ da\rho

$$F(\varphi) = A' \sin(m\varphi) + B' \cos(m\varphi)$$

$$\text{en } \textcircled{2} \quad \text{oplossen of } \text{oplyn Bessel: } r^2 R'' + r R' + (1^2 r^2 - m^2) R = 0 \text{ da}\rho \kappa = 1^2$$

$$\text{da}\rho R(r) = C' J_m(\lambda r) + D' Y_m(\lambda r) \text{ en } \text{da}\rho r=0, Y_m \rightarrow \infty \text{ da}\rho D'=0$$

$$\text{da}\rho R(r) = C' J_m(\lambda r)$$

$$\text{van en } \textcircled{1}: T'' + C^2 \kappa T = 0 \text{ da}\rho T(t) = E \sin(\lambda t) + G \cos(\lambda t)$$

$$U(2, \varphi, t) = 0 \Rightarrow J_m(2) = 0 \Rightarrow 2 = S_{mn} = \lambda = \frac{S_{mn}}{2}$$

$$\text{da}\rho U(r, \varphi, t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} J_m\left(\frac{S_{mn}}{2}r\right) (A_{mn} \sin(m\varphi) + B_{mn} \cos(m\varphi)) (E_{mn} \sin\left(\frac{S_{mn}}{2}ct\right) + G_{mn} \cos\left(\frac{S_{mn}}{2}ct\right))$$

ano optinis Garbius

$$U(r, \varphi, t) = J_4\left(\frac{S_{45}}{2}r\right)(A_{45}\sin(4\varphi) + B_{45}\cos(4\varphi))\left(E_{45}\sin\left(\frac{S_{45}}{2}ct\right) + G_{45}\cos\left(\frac{S_{45}}{2}ct\right)\right) + \\ + J_7\left(\frac{S_{73}}{2}r\right)(A_{73}\sin(7\varphi) + B_{73}\cos(7\varphi))\left(E_{73}\sin\left(\frac{S_{73}}{2}ct\right) + G_{73}\cos\left(\frac{S_{73}}{2}ct\right)\right)$$

ano znr nprzr optinis Garbius: $G_{45}=0$ | $B_{73}=0$ | $A_{73}G_{73} \geq 3$

ano zr džspn: $E_{73}=0$ | $A_{45}=0$ | $B_{45}E_{45}\frac{S_{45}c}{2} \geq 1$

Apa zduis:

$$U(r, \varphi, t) = \frac{2}{cS_{45}} J_4\left(\frac{S_{45}}{2}r\right) \cos(4\varphi) \sin\left(\frac{S_{45}}{2}ct\right) + 3 J_7\left(\frac{S_{73}}{2}r\right) \sin(7\varphi) \cos\left(\frac{S_{73}}{2}ct\right)$$