STRICTLY SINGULAR NON-COMPACT DIAGONAL OPERATORS ON HI SPACES

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Abstract. We construct a Hereditarily Indecomposable Banach space $\mathfrak{X}_d$ with a Schauder basis $(e_n)_{n\in\mathbb{N}}$ on which there exist strictly singular non-compact diagonal operators. Moreover, the space $\mathcal{L}_{\rm diag}(\mathfrak{X}_d)$ of diagonal operators with respect to the basis $(e_n)_{n\in\mathbb{N}}$ contains an isomorphic copy of $\ell_\infty(\mathbb{N})$.

Contents

1. Introduction

In the present paper we study the structure of the diagonal operators on Hereditarily Indecomposable spaces having a Schauder basis. The class of Hereditarily Indecomposable (HI) Banach spaces was introduced in the early 90's by W.T. Gowers and B. Maurey [23] and led to the solution of many long standing open problems in Banach space theory. Since then the class of HI Banach spaces, as well as the spaces of bounded linear operators acting on them have been studied extensively.

We begin by recalling that an infinite dimensional Banach space X is HI provided no closed subspace Y of X is of the form $Y = Z \oplus W$ with both Z, W being of infinite dimension. For a Banach space X we shall use $\mathcal{L}(X)$ to denote the space of bounded linear operators $T : X \to X$, while the notation $\mathcal{S}(X)$, $\mathcal{K}(X)$ will stand for the ideals of strictly singular and compact operators on X respectively. As was shown by Gowers and Maurey ([23]), for a complex HI space X, every $T \in \mathcal{L}(X)$ takes the form $T = \lambda I + S$ with $\lambda \in \mathbb{C}$ and $S \in \mathcal{S}(X)$ (by I we shall always denote the identity operator). However, it is not true in general, that each $T \in \mathcal{L}(X)$, for a real HI Banach space X, can be written as $T = \lambda I + S$ with $\lambda \in \mathbb{R}$ and $S \in S(X)$; although this happens for the space X_{GM} of Gowers and Maurey [23]

2000 *Mathematics Subject Classification.* 46B28, 46B20, 46B03.
Key words and phrases. Hereditarily Indecomposable Banach space, diagonal operator, strictly singular operator, compact operator.

and for the asymptotic ℓ_1 HI space X_{AD} constructed by Argyros and Deliyanni [5] (for a proof see e.g. [13]). V. Ferenczi proved ([17]) that for every real HI space X, the quotient space $\mathcal{L}(X)/\mathcal{S}(X)$, is a division algebra isomorphic to $\mathbb{R}$, to $\mathbb{C}$, or to the quaternionic algebra $\mathbb{H}$. V. Ferenczi [18] has presented two real HI Banach spaces $X_{\mathbb{C}}$ and $X_{\mathbb{H}}$ with $\mathcal{L}(X_{\mathbb{C}})/\mathcal{S}(X_{\mathbb{C}})$ isomorphic to $\mathbb{C}$ and $\mathcal{L}(X_{\mathbb{H}})/\mathcal{S}(X_{\mathbb{H}})$ isomorphic to $\mathbb{H}$. A variety of spaces X with a prescribed algebra $\mathcal{L}(X)/\mathcal{S}(X)$ were provided by Gowers and Maurey in [24]. Although these spaces X are not HI, they do not contain any unconditional basic sequence, hence, Gowers' dichotomy ([21], [22]) yields that they are HI saturated. Argyros and Manoussakis [11], provided an unconditionally saturated Banach space X with the property that every $T \in \mathcal{L}(X)$ is of the form $T = \lambda I + S$ with $S \in \mathcal{S}(X)$.

The problem of the existence of strictly singular non-compact operators on HI spaces has been studied by several authors. The first result in this direction, due to Gowers ([20]), is an operator $T : Y \to X_{GM}$, for some subspace Y of the Gowers-Maurey space X_{GM}, such that T is not of the form $T = \lambda i_{Y,X} + K$ with K compact, where $i_{Y,X}$ is the canonical injection from Y into X. Several extensions of the above result have been given in [1], [2] and [29].

Argyros and Felouzis ([7]) using interpolation methods, provided examples of HI spaces on which there do exist strictly singular non-compact operators. G. Androulakis and Th. Schlumprecht [3] constructed a strictly singular non-compact operator $T : X_{GM} \to X_{GM}$, while G. Gasparis [19], constructed strictly singular non-compact operators in the reflexive asymptotic ℓ_1 HI space X_{AD} of Argyros and Deliyanni. K. Beanland has extended Gasparis' result in the class of asymptotic ℓ_p HI spaces, for $1 < p < \infty$, in [14].

The structure of $\mathcal{L}(X)$ has been also studied for non-reflexive HI spaces ([13], [4], [27]). It is notable that in all these examples, each strictly singular operator $T \in \mathcal{L}(X)$ is a weakly compact one. It is an open problem whether there exists an HI Banach space X and $T \in \mathcal{L}(X)$ which is strictly singular and not weakly compact.

The scalar plus compact problem was recently solved by S. Argyros and R. Haydon [8]. It is shown that there exists an HI ℓ_1 predual Banach space $\mathfrak{X}_K$ such that every $T : \mathfrak{X}_K \to \mathfrak{X}_K$ is of the form $T = \lambda I + K$, with K a compact operator. The corresponding problem for reflexive spaces remains open.

The present paper is devoted to the study of the subalgebra of diagonal operators of a HI space X with a Schauder basis $(e_n)_{n\in\mathbb{N}}$. Let' s recall that for a Banach space X with an a priori fixed basis $(e_n)_{n\in\mathbb{N}}$, a bounded linear operator $T : X \to X$ is said to be diagonal, if for each n, Te_n is a scalar multiple of e_n, $Te_n = \lambda_n e_n$. We denote by $\mathcal{L}_{\text{diag}}(X)$ the space of all diagonal operators $T : X \to X$. Note that if the diagonal operator T is strictly singular then the sequence $(\lambda_n)_{n\in\mathbb{N}}$ of eigenvalues of T converges to 0.

As is well known, when the basis $(e_n)_{n\in\mathbb{N}}$ of the space X is an unconditional one, the space $\mathcal{L}_{\text{diag}}(X)$ is isomorphic to $\ell_\infty(\mathbb{N})$ and operator $T \in \mathcal{L}_{\text{diag}}(X)$ is strictly singular if and only if T is compact and this happens if and only if the sequence $(\lambda_n)_{n\in\mathbb{N}}$ of eigenvalues of T is a null sequence.

The following question arises naturally.

(Q) Do there exist strictly singular non-compact diagonal operators on some HI space with a Schauder basis?

The aim of the present paper is to give a positive answer to (**Q**), by defining a HI space $\mathfrak{X}_d$ with a basis, on which there exist strictly singular non-compact diagonal operators. More precisely the space $\mathcal{L}_{\text{diag}}(\mathfrak{X}_d)$ contains isomorphic copies of $\ell_\infty(\mathbb{N})$ in a natural manner.

It is worth pointing out that the construction of strictly singular non-compact diagonal operators lies heavily on the conditional structure of the underlying space $\mathfrak{X}_d$. Previous constructions, like [3], [19], concern the existence of strictly singular non-compact operators acting on the unconditional frame of the HI spaces. In particular Gasparis ([19]) based his construction on an elegant idea which allowed him to define a mixed Tsirelson space $T[(\mathcal{S}_{n_j}, \frac{1}{m_j})_j]$ such that its dual $T^*[(\mathcal{S}_{n_j}, \frac{1}{m_j})_j]$ admits a c_0^ω spreading model. An adaptation of Gasparis method in the frame of $T[(\mathcal{A}_{n_j}, \frac{1}{m_j})_j]$ is the first of the fundamental ingredients in our construction. More precisely, for an appropriate double sequence $(m_j, n_j)_j$, it it shown that the dual space $T^*[(\mathcal{A}_{n_j}, \frac{1}{m_j})_j]$ admits a c_0 spreading model. It is not known whether each mixed Tsirelson space of the form $T[(\mathcal{A}_{n_j}, \frac{1}{m_j})_j]$ not containing any $\ell_p(\mathbb{N})$ or $c_0(\mathbb{N})$, shares the aforementioned property. This problem remains open even for Schlumprecht' s space $S = T[(\mathcal{A}_n, \frac{1}{\log_2(n+1)})_n]$ ([28]). As follows from [26], the space S admits a ℓ_1 spreading model. This, however, does not guarantee the existence of a c_0 spreading model in S^*.

The second ingredient of our construction, is the finite block representability of the space J_{T_0} in every block subspace of $\mathfrak{X}_d$. The space J_{T_0}, defined in [10], has a Schauder basis $(t_n)_{n\in\mathbb{N}}$ which is conditional and dominates the summing basis of c_0. We shall discuss in more detail the above two ingredients in the rest of the introduction.

In section 2 we define a mixed Tsirelson space $T_0 = T[(\mathcal{A}_{n_j}, \frac{1}{m_j})_{j=1}^\infty]$ with an unconditional basis, such that its dual space T_0^* admits a c_0 spreading model. The space T_0 will be the unconditional frame required for the definition of the HI space $\mathfrak{X}_d$, in a similar manner as Schlumprecht's space [28] is the unconditional frame for the space X_{GM} of Gowers and Maurey [23] and as the asymptotic ℓ_1 space X_{ad} having an unconditional basis is the unconditional frame for the asymptotic ℓ_1 HI space X_{AD} [5]. The sequence $(m_j)_{j\in\mathbb{N}}$ we use for the space T_0, as well as for the space $\mathfrak{X}_d$, is inspired by Gasparis work ([19]) and is defined recursively as follows

$$m_1 = m_2 = 2, \qquad \text{and} \qquad m_j = m_{j-1}^2 = m_1 \cdot m_2 \cdot \ldots \cdot m_{j-1} \quad \text{for } j \geq 3,$$

while we require that the sequence $(n_j)_{j\in\mathbb{N}}$ increases rather fast, namely

$$n_1 \geq 2^3 m_3 \qquad \text{and} \qquad n_j \geq (4n_{j-1})^5 \cdot m_j \quad \text{for } j \geq 2.$$

As it is well known, the norm of the space $T_0 = T[(\mathcal{A}_{n_j}, \frac{1}{m_j})_{j=1}^\infty]$ satisfies the implicit formula

$$\|x\| = \max\{\|x\|_\infty, \sup_j \|x\|_j|\}$$

where $\|x\|_j = \frac{1}{m_j} \sup \sum\limits_{k=1}^{n_j} \|E_k x\|$ with the supremum taken over all families $(E_k)_{k=1}^{n_j}$ of successive finite sets. Note that the Schauder basis $(e_l)_{l\in\mathbb{N}}$ of T_0 is subsymmetric and also each $\| \, \|_j$ is an equivalent norm on T_0.

The fundamental property of mixed Tsirelson spaces, like the above T_0, is a biorthogonality described as follows. There exists a null sequence $(\varepsilon_i)_i$ of positive numbers, such that for every infinite dimensional subspace Z and every $j \in \mathbb{N}$, there exists a vector $z \in Z$ with $\|z_j\| = \|z_j\|_j$ and $\|z_j\|_i \leq \varepsilon_{\min\{i,j\}}$. A transparent example of this phenomenon are the vectors of the form $y_j = \frac{m_j}{n_j} \sum_{l=1}^{n_j} e_l$ in T_0, satisfying the following properties. $\|y_j\| = \|y_j\|_j = 1$ while $\|y_j\|_i \leq \frac{2}{m_i}$ for $i < j$ and $\|y_j\|_i \leq \frac{m_j}{m_i}$ for $i > j$.

As follows from Gasparis method the above unique evaluation of the vectors $(y_j)_j$ is no longer true for all averages of the basis. More precisely setting $p_j = n_1 \cdot n_2 \cdot \ldots \cdot n_{j-1}$ the following holds.

Proposition 1.1. For every $j \geq 3$ we have that

$$\|\frac{1}{p_j} \sum_{l=1}^{p_j} e_l\| \leq \frac{4}{m_j}.$$

As $p_j = \prod_{i=1}^{j-1} n_i$ and $m_j = \prod_{i=1}^{j-1} m_i$, it is easily shown that $\frac{1}{m_j} \leq \|\frac{1}{p_j} \sum_{l=1}^{p_j} e_l\|_i$ for $1 \leq i \leq j$. Hence we conclude that, unlikely for the vectors $\frac{m_j}{n_j} \sum_{l=1}^{n_j} e_l$, the vectors $\frac{m_j}{p_j} \sum_{l=1}^{p_j} e_l$ have simultaneous evaluations by the family of norms $(\| \ \|_i)_{1 \leq i \leq j}$. This actually yields that if we consider successive functionals $(\phi_j)_{j=3}^{\infty}$ of the form $\phi_j = \frac{1}{m_j} \sum_{l \in F_j} e_l^*$ with $\#(F_j) = p_j$, then the sequence $(\phi_j)_{j=3}^{\infty}$ generates a c_0 spreading model in T_0^*. The proof of Proposition 1.1 is more involved than the corresponding one for the vectors $\frac{1}{n_j} \sum_{l=1}^{n_j} e_l$ and requires some new techniques which could be of independent interest.

The existence of a sequence generating a c_0 spreading model in the dual space T_0^* is the basic tool for constructing strictly singular non-compact operators on T_0. This follows from the next general statement which is presented in Proposition 3.1 of section 3.

Proposition 1.2. Let X, Y be a pair of Banach spaces such that

(i) There exists a sequence $(x_n^*)_{n \in \mathbb{N}}$ in X^* generating a c_0 spreading model.
(ii) The space Y has a normalized Schauder basis $(e_n)_{n \in \mathbb{N}}$ and there exists a norming set D of Y (i.e. $D \subset Y^*$ and $\|y\| = \sup\{f(y) : \ f \in D\}$ for every $y \in Y$), such that for every $\varepsilon > 0$ there exists $M_\varepsilon \in \mathbb{N}$ such that for every $f \in D$,
$$\#\{n \in \mathbb{N} : \ |f(e_n)| > \varepsilon\} \leq M_\varepsilon.$$

Then there exists a strictly increasing sequence of integers $(q_n)_{n \in \mathbb{N}}$ such that the operator $T : X \to Y$ defined by the rule

$$T(x) = \sum_{n=1}^{\infty} x_{q_n}^*(x) e_n$$

is bounded and non-compact.

The fact that every mixed Tsirelson space of the form $T[(A_{n_j}, \frac{1}{m_j})_j]$ satisfies condition (ii) of the above proposition, yields that there exist strictly singular non-compact operators $S : T_0 \to T_0$.

In section 4, the space $\mathfrak{X}_d$ is defined with the use of the above defined sequences $(m_j)_{j\in\mathbb{N}}$, $(n_j)_{j\in\mathbb{N}}$. The norming set K_d of the space $\mathfrak{X}_d$ is defined to be the minimal subset of $c_{00}(\mathbb{N})$ such that:

(i) It contains $\{\pm e_n^*:\ n\in\mathbb{N}\}$.
(ii) It is symmetric and closed under the restriction of its elements on intervals of $\mathbb{N}$.
(iii) For each j, it is closed under the $(\mathcal{A}_{n_j},\frac{1}{m_j})$ operation.
(iv) For each $j\geq 2$, it closed under the $(\mathcal{A}_{n_{2j-1}},\frac{1}{\sqrt{m_{2j-1}}})$ operation on n_{2j-1} special sequences.

The special sequences are defined in the standard manner with the use of a Gowers-Maurey type coding function σ. Notice that, since $m_{j+1}=m_j^2$ for $j\neq 1$, condition (iv) is equivalent to saying that the set K_d is closed under the $(\mathcal{A}_{n_{2j+1}},\frac{1}{m_{2j}})$ operation on n_{2j+1} special sequences for each j. Using the standard methods for this purpose, we prove that the space $\mathfrak{X}_d$ is HI.

In section 5, a class of bounded diagonal operators on the space $\mathfrak{X}_d$ is defined. These diagonal operators are of the form $\sum\limits_k \lambda_k D_{j_k}$, where $(D_{j_k})_k$ is a sequence of diagonal operators with successive finite dimensional ranges. To be more precise, for each j and every choice of successive intervals $(I_i^j)_{i=1}^{p_j}$ we define a diagonal operator $D_j:\mathfrak{X}_d\to\mathfrak{X}_d$, by the rule

$$D_j(x)=\frac{1}{m_j}\sum_{i=1}^{p_j} I_i^j x.$$

Under certain growth conditions on the set $\{j_k:\ k\in\mathbb{N}\}$, we prove that for every $(\lambda_k)_{k\in\mathbb{N}}\in\ell_\infty(\mathbb{N})$ the diagonal operator $D=\sum\limits_k\lambda_k D_{j_k}:\mathfrak{X}_d\to\mathfrak{X}_d$ is bounded with $\|D\|\leq C_0\cdot\sup\limits_k|\lambda_k|$ for some universal constant C_0. It easily follows that such an operator D is strictly singular, since the space $\mathfrak{X}_d$ is HI and $\lim\limits_n D(e_n)=0$ (Proposition 1.2 of [13]).

In order to construct strictly singular non-compact diagonal operators on $\mathfrak{X}_d$ we prove that for appropriate choice of the intervals $\big((I_i^{j_k})_{i=1}^{p_{j_k}}\big)_{k=1}^\infty$ the corresponding diagonal operator $\sum\limits_k D_{j_k}$ is non-compact. The main tool for studying the structure of the space of diagonal operators on $\mathfrak{X}_d$, is the finite block representability of J_{T_0} in every block subspace of $\mathfrak{X}_d$. The space J_{T_0} is the Jamesification of the space T_0 described earlier. This class of spaces was defined by S. Bellenot, R. Haydon and E. Odell in [15]. Using the language of mixed Tsirelson spaces, we may write

$$J_{T_0}=T\big[G,\big(\mathcal{A}_{n_j},\frac{1}{m_j}\big)_{n\in\mathbb{N}}\big],$$

with $G=\{\pm\chi_I:\ I \text{ finite interval of } \mathbb{N}\}$. We prove that for every $N\in\mathbb{N}$ and every block subspace Z of $\mathfrak{X}_d$, there exists a block sequence $(z_k)_{k=1}^N$ in Z such that

$$\|\sum_{k=1}^N\mu_k t_k\|_{J_{T_0}}\leq\|\sum_{k=1}^N\mu_k z_k\|_{\mathfrak{X}_d}\leq c\cdot\|\sum_{k=1}^N\mu_k t_k\|_{J_{T_0}} \tag{1}$$

for c a universal constant. The notation $(t_n)_{n\in\mathbb{N}}$ stands for the standard basis of J_{T_0}. A similar result in a different context, is given by S. Argyros, J. Lopez-Abad and S. Todorcevic in [9], [10]. The precise definition of the space J_{T_0} is given in section 6, where the theorem of the finite block representability of J_{T_0} in every block subspace of $\mathfrak{X}_d$ is stated, postponing its proof for section 7. Section 6 is mainly devoted to the construction of the diagonal strictly singular non-compact operators on the space $\mathfrak{X}_d$.

For a given block subspace Z of $\mathfrak{X}_d$, using (1) in conjunction with some easy estimates on the basis of J_{T_0}, we construct successive block sequences $(y_k^j)_{k=1}^{2p_j}$ in Z, such that

$$(2) \qquad \|\frac{1}{2p_j}\sum_{k=1}^{p_j} y_{2k-1}^j\| \geq \frac{1}{2} \qquad \text{and} \qquad \|\frac{1}{2p_j}\sum_{k=1}^{2p_j}(-1)^{k+1}y_k^j\| \leq \frac{4c}{m_j}.$$

We set $D_j(x) = \frac{1}{m_j}\sum\limits_{i=1}^{p_j} I_i^j x$ for each j, where $I_i^j = \text{ran}(y_{2i-1}^j)$. Let' s point out that the diagonal operator D_j acting on the vector $x_j = \frac{m_j}{2p_j}\sum\limits_{i=1}^{2p_j}(-1)^{i+1}y_i^j$, ignores y_i^j when i is even. This in conjunction with (2) yields that $\|x_j\| \leq 4c$, $\|D_j x_j\| \geq \frac{1}{2}$. For a suitable choice of the set $\{j_k : \ k \in \mathbb{N}\}$, the diagonal operator $D = \sum\limits_k D_{j_k}$ is bounded and strictly singular, while it is non-compact (even the restriction of D on the subspace Z is non-compact) since for the block sequence $(x_{j_k})_{k\in\mathbb{N}}$ we have that $\|x_{j_k}\| \leq 4c$ while $\|Dx_{j_k}\| \geq \frac{1}{2}$.

Moreover, it is easily shown that for every $(\lambda_k)_{k\in\mathbb{N}} \in \ell_\infty(\mathbb{N})$,

$$\frac{1}{8c}\cdot\sup_k|\lambda_k| \leq \|\sum_{k=1}^{\infty}\lambda_k D_{j_k}\| \leq C_0\cdot\sup_k|\lambda_k|$$

hence the space $\mathcal{L}_{\text{diag}}(\mathfrak{X}_d)$ of diagonal operators of $\mathfrak{X}_d$ contains an isomorphic copy of $\ell_\infty(\mathbb{N})$. The next theorem summarizes the basic properties of the space $\mathfrak{X}_d$.

Theorem 1.3. There exists a Banach space $\mathfrak{X}_d$ with a Schauder basis $(e_n)_{n\in\mathbb{N}}$ such that:

(i) The space $\mathfrak{X}_d$ is reflexive and HI.
(ii) For every infinite dimensional subspace Z of $\mathfrak{X}_d$ there exists a diagonal strictly singular operator $D : \mathfrak{X}_d \to \mathfrak{X}_d$ such that the restriction of D on the subspace Z is a non-compact operator.
(iii) The space $\mathcal{L}_{\text{diag}}(\mathfrak{X}_d)$ of diagonal operators of $\mathfrak{X}_d$ with respect to the basis $(e_n)_{n\in\mathbb{N}}$ contains an isomorphic copy of $\ell_\infty(\mathbb{N})$.

As we have mentioned above the scalar plus compact problem remains open within the class of separable reflexive Banach spaces. Even the weaker problem related to the present work, namely the existence of a reflexive Banach space with a Schauder basis such that every diagonal operator is of the form $\lambda I + K$, with K a compact diagonal operator, is still open. In a forthcoming paper [6], we shall present a quasireflexive Banach space $\mathfrak{X}_D$ with a Schauder basis, such that the space $\mathcal{L}_{\text{diag}}(\mathfrak{X}_D)$ is HI and satisfies the scalar plus compact property.

2. The mixed Tsirelson space $T_0 = T[(\mathcal{A}_{n_j}, \frac{1}{m_j})_{j=1}^{\infty}]$

This section is devoted to the construction of a mixed Tsirelson space T_0 with an unconditional basis, such that the dual space T_0^* admits a sequence which generates a c_0 spreading model. This space is of the form $T_0 = T[(\mathcal{A}_{n_j}, \frac{1}{m_j})_{j\in\mathbb{N}}]$ with a very careful choice of the sequence $(m_j)_{j\in\mathbb{N}}$.

Notation 2.1. For a finite set F, we denote by $\#F$ the cardinality of the set F. We denote by $\mathcal{A}_n$ the class of subsets of $\mathbb{N}$ with cardinality less than or equal to n,

$$\mathcal{A}_n = \{F \subset \mathbb{N} : \#F \leq n\}.$$

By $c_{00}(\mathbb{N})$ we denote the vector space of all finitely supported sequences of reals and by either $(e_i)_{i=1}^{\infty}$ or $(e_i^*)_{i=1}^{\infty}$, depending on the context, its standard Hamel basis. For $x = \sum_{i=1}^{\infty} a_i e_i \in c_{00}(\mathbb{N})$, the support of x is the set $\operatorname{supp} x = \{i \in \mathbb{N} : \ a_i \neq 0\}$ while the range $\operatorname{ran} x$ of x, is the smallest interval of $\mathbb{N}$ containing $\operatorname{supp} x$. For nonempty finite subsets E, F of $\mathbb{N}$, we write $E < F$ if $\max E < \min F$. For $n \in \mathbb{N}$, $E \subset \mathbb{N}$ we write $n < E$ (resp. $n \leq E$) if $n < \min E$ (resp. $n \leq \min E$). For x, y nonzero vectors in $c_{00}(\mathbb{N})$, $x < y$ means $\operatorname{supp} x < \operatorname{supp} y$. For $n \in \mathbb{N}$, $x \in c_{00}(\mathbb{N})$, we write $n < x$ (resp. $n \leq x$) if $n < \operatorname{supp} x$ (resp. $n \leq \operatorname{supp} x$). We shall call the subsets $(E_i)_{i=1}^n$ of $\mathbb{N}$ successive if $E_1 < E_2 < \cdots < E_n$. Similarly, the vectors $(x_i)_{i=1}^n$ are called successive, if $x_1 < x_2 < \cdots < x_n$. For $x = \sum_{i=1}^{\infty} a_i e_i$ and E a subset of $\mathbb{N}$, we denote by Ex the vector $Ex = \sum_{i\in E} a_i e_i$. Finally, for $f = \sum_{i=1}^{\infty} \beta_i e_i^* \in c_{00}(\mathbb{N})$ and $x = \sum_{i=1}^{\infty} a_i e_i \in c_{00}(\mathbb{N})$ we denote by $f(x)$ the real number $f(x) = \sum_{i=1}^{\infty} a_i \beta_i$.

Definition 2.2. Let $n \in \mathbb{N}$ and $\theta \in (0, 1)$.

(i) A finite sequence $(f_i)_{i=1}^k$ in $c_{00}(\mathbb{N})$ is said to be $\mathcal{A}_n$ admissible if $k \leq n$ and $f_1 < f_2 < \cdots < f_k$.
(ii) The $(\mathcal{A}_n, \theta)$ operation on $c_{00}(\mathbb{N})$ is the operation which assigns to each $\mathcal{A}_n$ admissible sequence $f_1 < f_2 < \cdots < f_k$ the vector $\theta(f_1 + f_2 + \cdots + f_k)$.

Definition 2.3. Given a pair $(m_j)_{j\in I}$, $(n_j)_{j\in I}$ of either finite ($I = \{1, \ldots, k\}$) or infinite ($I = \mathbb{N}$) increasing sequences of integers we shall denote by $K = K[(m_j, n_j)_{j\in I}]$ the minimal subset of $c_{00}(\mathbb{N})$ satisfying the following conditions.

(i) $\{\pm e_i^* : \ i \in \mathbb{N}\} \subset K$.
(ii) For each $j \in I$, K is closed under the $(\mathcal{A}_{n_j}, \frac{1}{m_j})$ operation.

It is easy to check that the set K is symmetric and closed under the restriction of its elements on subsets of $\mathbb{N}$.

Let $j \in \mathbb{N}$. If $f \in K$ is the result of the $(\mathcal{A}_{n_j}, \frac{1}{m_j})$ operation on some sequence $f_1 < f_2 < \cdots < f_k$ ($k \leq n_j$) in K, we shall say that the weight of f is m_j and we shall denote this fact by $w(f) = m_j$. We note however that the weight $w(f)$ of a functional $f \in K$ is not necessarily uniquely determined.

Definition 2.4. [The tree T_f of a functional $f \in K$] Let $f \in K$. By a tree of f (or tree corresponding to the analysis of f) we mean a finite family $T_f = (f_a)_{a\in\mathcal{A}}$ indexed by a finite tree $\mathcal{A}$ with a unique root $0 \in \mathcal{A}$ such that the following conditions are satisfied:

(i) $f_0 = f$ and $f_a \in K$ for all $a \in \mathcal{A}$.
(ii) If a is maximal in $\mathcal{A}$, then $f_a = \pm e_k^*$ for some $k \in \mathbb{N}$.

(iii) For every $a \in \mathcal{A}$ which is not maximal denoting by S_a the set of immediate successors of a in $\mathcal{A}$ the following holds. There exists $j \in \mathbb{N}$ such that the family $(f_\beta)_{\beta \in S_a}$ is $\mathcal{A}_{n_j}$ admissible and $f_a = \frac{1}{m_j} \sum\limits_{\beta \in S_a} f_\beta$. In this case we say that $w(f_a) = m_j$.

The order $o(f_a)$ for each $a \in \mathcal{A}$ is also defined by backward induction as follows. If $f_a = \pm e_k^*$ then $o(f_a) = 1$, while if $f_a = \frac{1}{m_j} \sum\limits_{\beta \in S_a} f_\beta$ then $o(f_a) = 1 + \max\{o(f_\beta :$ $\beta \in S_a\}$. The order $o(T_f)$ of the aforementioned tree is defined to be equal to $o(f_0)$ (where $0 \in \mathcal{A}$ is the unique root of the tree $\mathcal{A}$).

Remark 2.5. An easy inductive argument yields the following.

(i) Every $f \in K$ admits a tree, not necessarily unique.
(ii) For every $\phi \in K$, if $\operatorname{supp}(\phi) = \{k_1 < k_2 < \cdots < k_d\}$ then for every $l_1 < l_2 < \cdots < l_d$ in $\mathbb{N}$ the functional $\psi = \sum\limits_{i=1}^{d} \phi(e_{k_i}) e_{l_i}^*$ also belongs to the set K.
(iii) For every $\phi \in K$ and every $E \subset \mathbb{N}$ the functional $E\phi$ also belongs to the set K.
(iv) If $\phi = \sum\limits_{i=1}^{\infty} a_i e_i \in K$, then for every choice of signs $(\varepsilon_i)_{i=1}^{\infty}$ the functional $\sum\limits_{i=1}^{\infty} \varepsilon_i a_i e_i$ also belongs to K.

Definition 2.6. The order $o(f)$ of an $f \in K$, is defined as

$$o(f) = \min\{o(T_f) : \ T_f \text{ is a tree of } f\}.$$

In general, given a symmetric subset W of $c_{00}(\mathbb{N})$ containing $\{\pm e_k^* : \ k \in \mathbb{N}\}$, the norm induced by W on $c_{00}(\mathbb{N})$ is defined as follows. For every $x \in c_{00}(\mathbb{N})$,

$$\|x\|_W = \sup\{f(x) : \ f \in W\}.$$

In the case where $W = K = K[(m_j, n_j)_{j \in I}]$ for a given double sequence $(m_j, n_j)_{j \in I}$, the completion of the corresponding normed space $(c_{00}(\mathbb{N}), \|\cdot\|_K)$ is denoted by $T[(\mathcal{A}_{n_j}, \frac{1}{m_j})_{j \in I}]$ and is called the mixed Tsirelson space defined by the family $(\mathcal{A}_{n_j}, \frac{1}{m_j})_{j \in I}$. The norming set K is called the standard norming set of the space $T[(\mathcal{A}_{n_j}, \frac{1}{m_j})_{j \in I}]$.

Remark 2.7. (i) As follows from Remark 2.5(iii), (iv), the Hamel basis $(e_i)_{i \in \mathbb{N}}$ of $c_{00}(\mathbb{N})$ is a 1-unconditional Schauder basis for the space $T[(\mathcal{A}_{n_j}, \frac{1}{m_j})_{j \in I}]$.
(ii) If $x \in c_{00}(\mathbb{N})$ with $\operatorname{supp} x = \{k_1 < k_2 < \cdots < k_d\}$ and $l_1 < l_2 < \cdots < l_d$ then the vector $y = \sum\limits_{i=1}^{d} e_{k_i}^*(x) e_{l_i}$ satisfies $\|x\|_K = \|y\|_K$, thus the basis $(e_i)_{i \in \mathbb{N}}$ is subsymmetric. This is also a consequence of Remark 2.5.

For the definition of the space T_0 and of the Hereditarily Indecomposable space $\mathfrak{X}_d$ later, we shall use a specific choice of the sequences $(m_j)_{j \in \mathbb{N}}$,$(n_j)_{j \in \mathbb{N}}$ described in the next definition. In the sequel $(m_j)_{j \in \mathbb{N}}$, $(n_j)_{j \in \mathbb{N}}$ will always stand for these sequences.

Definition 2.8. [The sequences $(m_j)_{j\in\mathbb{N}}$,$(n_j)_{j\in\mathbb{N}}$ and the space T_0]
We set $m_1 = m_2 = 2$, and for $j \geq 3$ we define

$$m_j = m_{j-1}^2 = \prod_{i=1}^{j-1} m_i.$$

We choose a sequence $(n_j)_{j=1}^{\infty}$ as follows: $n_1 \geq 2^3 m_3$, and for every $j \geq 2$ we choose

$$n_j \geq (4n_{j-1})^5 \cdot m_j$$

Observe, for later use, that $n_j \geq 2^{j+2} m_{j+2}$ while, setting $p_j = n_1 \cdot n_2 \cdot \ldots \cdot n_{j-1}$, we have that $n_j \geq j p_j$. We notice here that the numbers $(p_j)_{j\geq 3}$ will play a key role in our proofs.

We set

$$T_0 = T\left[\left(\mathcal{A}_{n_j}, \frac{1}{m_j}\right)_{j=1}^{\infty}\right]$$

and we denote by K_0 the standard norming set of T_0.

Our aim is to prove that T_0^* has a block sequence which generates a c_0 spreading model (Proposition 2.13). The main step of the proof is done in Lemma 2.10. For its proof we need to recall the definition of the modified Tsirelson spaces $T_M[(\mathcal{A}_n, \theta_n)_{n\in I}]$. For a given (finite or infinite) subset I of $\mathbb{N}$ and a sequence $(\theta_n)_{n\in I}$ in $(0,1)$, with $\lim\limits_{n\in I, n\to\infty} \theta_n = 0$ if I is infinite, the set $K_M = K_M[(\mathcal{A}_n, \theta_n)_{n\in I}]$ is defined as follows:

The set K_M is the minimal subset of $c_{00}(\mathbb{N})$ with the following properties:

(i) $\{\pm e_k^* : k \in \mathbb{N}\} \subset K_M$.
(ii) For every $n \in I$, every $m \leq n$ and every sequence $(\phi_k)_{k=1}^m$ in K_M with pairwise disjoint supports, we have that $\theta_n\big(\sum\limits_{k=1}^m \phi_k\big) \in K_M$.

We define the norm $\|\cdot\|_M$ on $c_{00}(\mathbb{N})$ by the rule

$$\|x\|_M = \sup\{\phi(x) : \ \phi \in K_M\}$$

for every $x \in c_{00}(\mathbb{N})$. The space $T_M[(\mathcal{A}_n, \theta_n)_{n\in I}]$ is the completion of the space $(c_{00}(\mathbb{N}), \|\cdot\|_M)$.

It is proved in [16] that a space of the form $X = T[(\mathcal{A}_{n_i}, \frac{1}{m_i})_{i=1}^k]$ is isomorphic to $\ell_p(\mathbb{N})$ for some $1 < p < \infty$ (or $c_0(\mathbb{N})$). Under the condition that the sequence $\big(\log_{m_i}(n_i)\big)_{i=1}^k$ is increasing (which is satisfied by the sequences (m_i) and (n_i) used in the definition of T_0) this p is the conjugate exponent of $q = \log_{m_k}(n_k)$. In particular, it is shown in [16] that, for every $f \in c_{00}(\mathbb{N})$, we have $\|f\|_q \leq \|f\|_{X^*}$, where $\|\cdot\|_q$ denotes the norm of $\ell_q(\mathbb{N})$.

Using the same argument (induction and Hölder's inequality) one can also get the inequality $\|f\|_q \leq \|f\|_{X_M^*}$ where $\|\cdot\|_{X_M^*}$ is the norm of the dual of the modified space $X_M = T_M[(\mathcal{A}_{n_i}, \frac{1}{m_i})_{i=1}^k]$. We note for completeness that, using the obvious inequality $\|f\|_{X_M^*} \leq \|f\|_{X^*}$, we get that in fact X_M is isomorphic to X (and $\ell_p(\mathbb{N})$).

Lemma 2.9. Let $j \in \mathbb{N}$, $j \geq 3$. We denote by $K_M(j-2)$ the norming set of the modified space $T_M[(\mathcal{A}_{n_i}, \frac{1}{m_i})_{i=1}^{j-2}]$. Let $\phi \in K_M(j-2)$ be such that, for every $l \in \text{supp}(\phi)$, we have that $\phi(e_l) > \frac{1}{m_j}$. Then,

$$\#\,\text{supp}(\phi) \leq n_{j-1}.$$

Proof. For the space $X_M = T_M[(\mathcal{A}_{n_i}, \frac{1}{m_i})_{i=1}^{j-2}]$ where $(m_i)_i$ and $(n_i)_i$ are as in the definition of T_0, the inequality $\|\phi\|_q \leq \|\phi\|_{X_M^*}$ with $q = \log_{m_{j-2}}(n_{j-2})$ implies the following: If $\phi \in B_{X_M^*}$ and $\phi(e_l) > \frac{1}{m_j}$ for every $l \in \mathrm{supp}(\phi)$, then

$$\frac{(\#\,\mathrm{supp}(\phi))^{1/q}}{m_j} < \|\phi\|_q \leq \|\phi\|_{X_M^*} \leq 1.$$

Since $m_j = m_{j-2}^4$ and $n_{j-2}^4 < n_{j-1}$, we get that $\#\,\mathrm{supp}(\phi) < m_j^q = (m_{j-2}^q)^4 = n_{j-2}^4 \leq n_{j-1}$. □

Lemma 2.10. Let $j \geq 3$ and let $k_1 < k_2 < \cdots < k_{p_j}$. Then

$$\|\frac{1}{p_j}\sum_{i=1}^{p_j} e_{k_i}\| \leq \frac{4}{m_j}.$$

Proof. From the subsymmetricity of the basis $(e_i)_{i\in\mathbb{N}}$ (Remark 2.7(ii)) it is enough to show that $\|\sum\limits_{l=1}^{p_j} e_l\| \leq \frac{4p_j}{m_j}$. Let $f \in K_0$; we shall show that $f(\sum\limits_{l=1}^{p_j} e_l) \leq \frac{4p_j}{m_j}$. We may assume that $f(e_l) \geq 0$ for all l (Remark 2.5(iv)).

We set $D = \{l \in \mathrm{supp}(f) :\ \phi(e_l) > \frac{1}{m_j}\}$ and we define $\phi = f|_D$ and $\psi = f|_{\mathbb{N}\setminus D}$. Since obviously $\psi(\sum\limits_{i=1}^{p_j} e_i) \leq \frac{p_j}{m_j}$ it is enough to show that $\phi(\sum\limits_{i=1}^{p_j} e_i) \leq \frac{3p_j}{m_j}$.

Fix a tree analysis $T_\phi = (\phi_a)_{a\in\mathcal{A}}$ of the functional ϕ. For every $l \in \mathrm{supp}(\phi)$ we define the set $A_l = \{i :\ \exists a \in \mathcal{A}$ with $l \in \mathrm{supp}(f_a)$ and $w(f_a) = m_i\}$ and for each $i \in A_l$ we denote by $d_{l,i}$ the cardinality of the set $\{a \in \mathcal{A} :\ l \in \mathrm{supp}(f_a)$ and $w(f_a) = m_i\}$. Then, for each $l \in \mathrm{supp}(\phi)$,

$$\prod_{i\in A_l} \frac{1}{m_i^{d_{l,i}}} = \phi(e_l) > \frac{1}{m_j}.$$

Thus we have that $\prod\limits_{i\in A_l} m_i^{d_{l,i}} < m_j$ which in conjunction to the fact that $d_{l,i} \geq 1$ for each $i \in A_l$ and taking into account that $m_j = m_1 \cdot m_2 \cdot \ldots \cdot m_{j-1}$ we get the following:

(1) A_l is a proper subset of $\{1, \ldots, j-1\}$.
(2) If $j-1 \in A_l$, then $d_{l,j-1} = 1$. In general, if $j-1, \ldots, j-k \in A_l$, then $d_{l,j-1} = d_{l,j-2} = \cdots = d_{l,j-k} = 1$.

We partition the set $\mathrm{supp}(\phi)$ in the sets $(B_i)_{i=1}^{j-1}$ defined as follows. We set

$$B_{j-1} = \{l \in \mathrm{supp}(\phi) :\ j-1 \notin A_l\},$$

and for $k = 2, \ldots, j-1$, we set

$$B_{j-k} = \{l \in \mathrm{supp}(\phi) :\ j-i \in A_l \text{ for } 1 \leq i < k \text{ and } j-k \notin A_l\}.$$

In the following three steps we estimate the action of ϕ on B_{j-1}, B_{j-2} and (in the general case) on B_{j-k}.

Step 1. The functional $\phi|_{B_{j-1}}$ satisfies the assumptions of Lemma 2.9, hence

$$|\phi(\sum_{l\in B_{j-1}} e_i)| \leq \#(\mathrm{supp}\,(\phi|_{B_{j-1}})) \leq n_{j-1}.$$

Step 2. Let $\phi' = \phi|_{B_{j-2}}$ and let $T_{\phi'} = (f_a)_{a\in\mathcal{A}'}$ be the restriction of the analysis T_ϕ on B_{j-2}. Then, for every $l \in \operatorname{supp}(\phi') = B_{j-2}$, there exists exactly one $a \in \mathcal{A}'$ such that $l \in \operatorname{supp}(f_a)$ and $w(f_a) = m_{j-1}$.

Claim. There exist disjointly supported functionals $(\phi_s)_{s=1}^{n_{j-1}}$ such that

$$\phi' = \frac{1}{m_{j-1}} \sum_{s=1}^{n_{j-1}} \phi_s$$

with $\phi_s \in K_M(j-3)$ for $1 \le s \le n_{j-1}$.

Proof of the Claim. Let

$$\mathcal{B} = \{a \in \mathcal{A}' : \ w(f_a) = m_{j-1}\}.$$

By the definition of ϕ', the functionals $(f_a)_{a\in\mathcal{B}}$, have pairwise disjoint supports and

$$\bigcup_{a\in\mathcal{B}} \operatorname{supp}(f_a) = \operatorname{supp}(\phi').$$

For each $a \in \mathcal{A}'$ we write

$$f_a = \frac{1}{m_{j-1}} \sum_{\beta\in S_a} f_\beta = \frac{1}{m_{j-1}} \sum_{k=1}^{n_{j-1}} f_{\widehat{ak}},$$

where $f_{\widehat{ak}} = 0$ if $\#S_a < k \le n_{j-1}$.

We now build the disjointly supported functionals $(\phi_s)_{s=1}^{n_{j-1}}$. We fix s and we define inductively the analysis $(f^s_\gamma)_{\gamma\in\mathcal{A}'}$ of ϕ_s as follows: Let $\gamma \in \mathcal{A}'$ be a maximal node, i.e. $f_\gamma = e^*_{k_\gamma}$. Then there exists a unique $a \in \mathcal{B}$ such that $a \prec \gamma$. If $\gamma = \widehat{as}$ or $\widehat{as} \prec \gamma$, then we set $f^s_\gamma = f_\gamma$. Otherwise, we set $f^s_\gamma = 0$.

Let now $\gamma \in \mathcal{A}'$, γ not maximal, with $f_\gamma = \frac{1}{m_r} \sum_{\beta\in S_\gamma} f_\beta$ and assume that f^s_β, $\beta \in S_\gamma$, have been defined. If $\gamma \notin \mathcal{B}$ then we set $f^s_\gamma = \frac{1}{m_r} \sum_{\beta\in S_\gamma} f^s_\beta$. If $\gamma \in \mathcal{B}$ then $f_\gamma = \frac{1}{m_{j-1}} \sum_{k=1}^{n_{j-1}} f_{\widehat{\gamma k}}$ and we set $f^s_\gamma = f_{\widehat{\gamma s}}$.

This completes the inductive construction. It is now easy to check that the functionals $\phi_s = f^s_0$, $s = 1, \ldots, n_{j-1}$, (recall that $0 \in \mathcal{A}$ is the unique root of the tree $\mathcal{A}$) have the desired properties and this completes the proof of the claim. $\square$

Since $\phi'(e_l) = \phi(e_l) > \frac{1}{m_j}$ for each $l \in \operatorname{supp}(\phi')$ it follows that for every s, $1 \le s \le n_{j-1}$ and every $l \in \operatorname{supp}(\phi_s)$, we have that

$$\phi_s(e_l) > \frac{m_{j-1}}{m_j} = \frac{1}{m_{j-1}}.$$

Thus, for every $s = 1, \ldots, n_{j-1}$, the functional ϕ_s satisfies the assumptions of Lemma 2.9, with j replaced by $j-1$, so

$$\#\operatorname{supp}(\phi_s) \le n_{j-2}.$$

It follows that

$$\phi\Big(\sum_{l\in B_{j-2}} e_l\Big) = \frac{1}{m_{j-1}} \sum_{s=1}^{n_{j-1}} \phi_s\Big(\sum_{l\in\operatorname{supp}(\phi_s)} e_l\Big) \le \frac{1}{m_{j-1}} n_{j-1} n_{j-2}.$$

Step 3. Let $3 \le k \le j-1$, set $\phi' = \phi|_{B_{j-k}}$, and let $T_{\phi'} = (f_a)_{a\in\mathcal{A}'}$ be the corresponding analysis. Then, for every $l \in \operatorname{supp}(\phi')$ and for every $i = 1, \ldots, k-1$, there exists exactly one $a \in \mathcal{A}'$ such that $l \in \text{supp}(f_a)$ and $w(f) = m_{j-i}$. As in Step 2, it follows by induction that we can write

$$\phi' = \frac{1}{m_{j-1}} \frac{1}{m_{j-2}} \cdots \frac{1}{m_{j-k+1}} \left(\sum_{s=1}^{n_{j-k+1} \cdot \ldots \cdot n_{j-1}} \phi_s \right),$$

where the functionals $(\phi_s)_{s=1}^{n_{j-k+1} \cdot \ldots \cdot n_{j-1}}$ have pairwise disjoint supports for $1 \leq s \leq n_{j-k+1} \cdot \ldots \cdot n_{j-1}$, $\phi_s \in K_M(j-k-1)$ while for every $l \in \text{supp}(\phi_s)$,

$$\phi_s(e_l) > \frac{m_{j-1} \cdot \ldots \cdot m_{j-k+1}}{m_j} = \frac{1}{m_{j-k+1}}.$$

For every s, the functional ϕ_s satisfies the assumptions of Lemma 2.9 with j replaced by $j-k+1$, so

$$\# \, \text{supp}(\phi_s) \leq n_{j-k}.$$

It follows that

$$\phi(\sum_{l \in B_{j-k}} e_l) \leq \frac{1}{m_{j-1} \cdot m_{j-2} \cdot \ldots \cdot m_{j-k+1}} \cdot (n_{j-1} \cdot n_{j-2} \cdot \ldots \cdot n_{j-k+1}) \cdot n_{j-k}.$$

We conclude that

$$\begin{aligned}
\phi(\sum_{l=1}^{p_j} e_l) &\leq \phi(\sum_{l \in B_{j-1}} e_l) + \cdots + \phi(\sum_{l \in B_1} e_l) \\
&\leq n_{j-1} + \frac{1}{m_{j-1}} n_{j-1} n_{j-2} + \cdots + \frac{1}{m_{j-1} \cdot \ldots \cdot m_{j-k+1}} n_{j-1} \cdot \ldots \cdot n_{j-k} \\
&\quad + \cdots + \frac{1}{m_{j-1} \cdot \ldots \cdot m_2} n_{j-1} \cdot \ldots \cdot n_1 \\
&= n_{j-1} + \frac{1}{m_j} \Big(\sum_{k=2}^{j-1} \frac{m_j}{m_{j-1} \cdot \ldots \cdot m_{j-k+1}} n_{j-1} \cdot \ldots \cdot n_{j-k} \Big) \\
&= \frac{1}{m_j} \Big(\sum_{k=1}^{j-1} m_{j-k+1} n_{j-1} \cdot \ldots \cdot n_{j-k} \Big) \\
&\quad \text{(using the property } n_i \geq 2^{i+2} m_{i+2}) \\
&\leq \frac{1}{m_j} \Big(\sum_{k=1}^{j-2} \frac{1}{2^{j-k+1}} n_{j-k-1} n_{j-k} \cdots n_{j-1} \Big) + \frac{m_2}{m_j} n_1 \cdot \ldots \cdot n_{j-1} \\
&\leq \frac{1}{m_j} \Big(\sum_{k=1}^{j-2} \frac{1}{2^{j-k+1}} \Big) p_j + \frac{2}{m_j} p_j \\
&\leq \frac{3 p_j}{m_j}.
\end{aligned}$$

The proof of the lemma is complete. ∎

Definition 2.11. We say that a sequence $(z_n)_{n \in \mathbb{N}}$ in a Banach space Z generates a c_0 spreading model provided that there exists a constant $C \geq 1$ such that for every $s \leq k_1 < k_2 < \cdots < k_s$, the finite sequence $(z_{k_i})_{i=1}^s$ is C equivalent to the standard basis of ℓ_∞^n.

Remark 2.12. A sequence $(z_n)_{n\in\mathbb{N}}$ generating a c_0 spreading model is necessarily weakly null. Indeed, assume the contrary. Then there exists $\varepsilon > 0$, $f \in Z^*$ and M an infinite sequence of $\mathbb{N}$ such that $f(z_n) \geq \varepsilon$ for all $n \in M$. Choose $s > \frac{C}{\varepsilon}$ (where C is the constant of the c_0 spreading model) and $s \leq k_1 < k_2 < \cdots < k_s$ with $k_i \in M$. Then from our assumption about the sequence $(z_n)_{n\in\mathbb{N}}$ we get that $\|z_{k_1} + z_{k_2} + \cdots + z_{k_s}\| \leq C$. On the other hand the action of the functional f yields that $\|z_{k_1} + z_{k_2} + \cdots + z_{k_s}\| \geq \sum\limits_{i=1}^{s} f(z_{k_i}) \geq s\varepsilon > C$, a contradiction.

Proposition 2.13. There exists a block sequence in T_0^* which generates a c_0 spreading model.

Proof. Let $(F_j)_{j=3}^{\infty}$ be a sequence of successive subsets of $\mathbb{N}$ with $\#F_j = p_j$, for each $j = 3, 4, \ldots$. For $j = 3, 4, \ldots$ we set

$$\phi_j = \frac{1}{m_j} \sum_{k\in F_j} e_k^*.$$

Then $\phi_j \in K_0$, thus $\|\phi_j\| \leq 1$, and

$$\phi_j\Big(\frac{1}{p_j} \sum_{k\in F_j} e_k\Big) = \frac{1}{m_j}.$$

From Lemma 2.10, we get that

$$\Big\|\frac{1}{p_j} \sum_{k\in F_j} e_k\Big\| \leq \frac{4}{m_j}.$$

It follows that, for every $j = 3, 4, \ldots$

$$\frac{1}{4} \leq \|\phi_j\| \leq 1.$$

We shall show that the sequence $(\phi_j)_{j=3}^{\infty}$ generates a c_0 spreading model. This is a direct consequence of the following:

Claim. For every $s \in \mathbb{N}$, $s \geq 3$, and every choice of indices $j_1 < j_2 < \cdots < j_s$ with $s \leq j_1$, the functional $\sum\limits_{k=1}^{s} \phi_{j_k}$ belongs to K_0.

Proof of the Claim. Fix s and $j_1 < j_2 < \cdots < j_s \in \mathbb{N}$ with $s \leq j_1$. For every $k = 2, 3, \ldots, s$, we write

$$\phi_{j_k} = \frac{1}{m_{j_k}} \sum_{i\in F_{j_k}} e_i^* = \frac{1}{m_{j_1}} \frac{1}{m_{j_1} \cdot m_{j_1+1} \cdot \ldots \cdot m_{j_k-1}} \sum_{i\in F_{j_k}} e_i^*.$$

Since

$$\#F_{j_k} = p_{j_k} = n_1 \cdot \ldots \cdot n_{j_1-1} \cdot n_{j_1} \cdot \ldots \cdot n_{j_k-1} = p_{j_1} \cdot (n_{j_1} \cdot n_{j_1+1} \cdot \ldots \cdot n_{j_k-1}),$$

we can partition the set F_{j_k} into p_{j_1} successive subsets $(G_l^k)_{l=1}^{p_{j_1}}$ where $\#G_l^k = n_{j_1} \cdot n_{j_1+1} \cdot \ldots \cdot n_{j_k-1}$ for every $l = 1, \ldots, p_{j_1}$.

Then, for every $l = 1, \ldots, p_{j_1}$, the functional

$$\psi_l^k = \frac{1}{m_{j_1} \cdot m_{j_1+1} \cdot \ldots \cdot m_{j_k-1}} \sum_{i\in G_l^k} e_i^*$$

belongs to K_0. It follows that, for every $k = 2, \dots, s$, we can write

$$\phi_{j_k} = \frac{1}{m_{j_1}} \sum_{l=1}^{p_{j_1}} \psi_l^k$$

where $\text{supp}(\psi_1^k) < \text{supp}(\psi_2^k) < \cdots < \text{supp}(\psi_{p_{j_1}}^k)$ and $\psi_l^k \in K_0$ for every $l = 1, \dots, p_{j_1}$. Since $s \leq j_1$ and $sp_{j_1} \leq j_1 p_{j_1} \leq n_{j_1}$, we get that the functional

$$\phi = \sum_{k=1}^{s} \phi_{j_k} = \frac{1}{m_{j_1}} \left(\sum_{i \in F_{j_1}} e_i^* + \sum_{k=2}^{s} \left(\sum_{l=1}^{p_{j_1}} \psi_l^k \right) \right)$$

belongs to K_0. This completes the proof of the Claim. □

The proof of the claim finishes, as we have mentioned earlier, the proof of the proposition. □

3. Strictly singular non-compact operators on T_0

The main step in other examples, where strictly singular non-compact operators are produced on Hereditarily Indecomposable Banach spaces (e.g. [3], [19]), is the contsruction of strictly singular non-compact operators on the mixed Tsirelson spaces which are the unconditional frames of those space. In this section, we show how the existence of a sequence generating a c_0 spreading model in T_0^* (Proposition 2.13), leads to strictly singular non-compact operators on T_0. In Proposition 3.1, which is of general nature, we prove how the existence of a c_0 spreading model in X^* leads to strictly singular non-compact operators $T : X \to Y$ for certain spaces Y, and then we apply this proposition to obtain the aforementioned result.

We also notice that it is not known whether each mixed Tsirelson space which is arbitrarily distortable admits a strictly singular non-compact operator.

Proposition 3.1. Let X, Y be a pair of Banach spaces such that

(i) There exists a sequence $(x_n^*)_{n\in\mathbb{N}}$ in X^* generating a c_0 spreading model.

(ii) The space Y has a normalized Schauder basis $(e_n)_{n\in\mathbb{N}}$ and there exists a norming set D of Y (i.e. $D \subset Y^*$ and $\|y\| = \sup\{f(y) : f \in D\}$ for every $y \in Y$), such that for every $\varepsilon > 0$ there exists $M_\varepsilon \in \mathbb{N}$ such that for every $f \in D$,

$$\#\{n \in \mathbb{N} : |f(e_n)| > \varepsilon\} \leq M_\varepsilon.$$

Then there exists a strictly increasing sequence of integers $(q_n)_{n\in\mathbb{N}}$ such that the operator $T : X \to Y$ defined by the rule

$$T(x) = \sum_{n=1}^{\infty} x_{q_n}^*(x) e_n$$

is bounded and non-compact.

Proof. Since the sequence $(x_n^*)_{n\in\mathbb{N}}$ generates a c_0 spreading model it is weakly null, hence, since it belongs to a dual space is also w^* null. From a result of W. B. Johnson and H. P. Rosenthal ([25]), passing to a subsequence we may assume that $(x_n^*)_{n\in\mathbb{N}}$ is a w^*- basic sequence. In particular there exists a bounded sequence $(x_n)_{n\in\mathbb{N}}$ in X such that $(x_n, x_n^*)_{n\in\mathbb{N}}$ are biorthogonal (i.e. $x_i^*(x_j) = \delta_{ij}$ for each i, j).

We select $(\theta_j)_{j\in\mathbb{N}}$ a strictly decreasing sequence of positive reals, with $\theta_1 = 1$, such that $\sum\limits_{j=1}^{\infty} j\theta_j < \infty$. From our assumption (ii) we may select a strictly increasing sequence $(q_n)_{n\in\mathbb{N}}$ in $\mathbb{N}$ such that for every $j \in \mathbb{N}$ and every $f \in D$,

$$\#\{n \in \mathbb{N} : \ |f(e_n)| > \theta_{j+1}\} \le q_j.$$

We claim that the operator $T : X \to Y$ defined by the rule $T(x) = \sum\limits_{n=1}^{\infty} x^*_{q_n}(x)e_n$ is bounded and non-compact.

We first show the boundedness of the operator T. Let $x \in X$ and $f \in D$. For each j we set

$$B_j = \{n \in \mathbb{N} : \ \theta_{j+1} < |f(e_n)| \le \theta_j\}.$$

From the definition of the sequence $(q_j)_{j\in\mathbb{N}}$ it follows that $\#(B_j) \le q_j$. We partition each set B_j in the following way:

$$C_j = \{n \in B_j : \ n \ge j\} \quad \text{and} \quad D_j = \{n \in B_j : \ n < j\}$$

Obviously $\#(D_j) \le j - 1$. Since also

$$\#\{q_n : \ n \in C_j\} = \#(C_j) \le \#(B_j) \le q_j \le \min\{q_n : \ n \in C_j\},$$

using our assumption (i), it follows that $\sum\limits_{n\in C_j} |x^*_{q_n}(x)| \le C\|x\|$, where C is the constant of the c_0 spreading model. Thus for each j,

$$\begin{aligned}\sum_{n\in B_j} |f(e_n)| \cdot |x^*_{q_n}(x)| &= \sum_{n\in C_j} |f(e_n)| \cdot |x^*_{q_n}(x)| + \sum_{n\in D_j} |f(e_n)| \cdot |x^*_{q_n}(x)| \\ &\le \theta_j C\|x\| + \theta_j(j-1)C\|x\| = j\theta_j C\|x\|.\end{aligned}$$

It follows that

$$\begin{aligned}|f(\sum_{n=1}^{\infty} x^*_{q_n}(x)e_n)| &\le \sum_{n=1}^{\infty} |f(e_n)| \cdot |x^*_{q_n}(x)| \\ &\le \sum_{j=1}^{\infty} \sum_{n\in B_j} |f(e_n)| \cdot |x^*_{q_n}(x)| \\ &\le C(\sum_{j=1}^{\infty} j\theta_j)\|x\|.\end{aligned}$$

Therefore the operator T is bounded with $\|T\| \le C(\sum\limits_{j=1}^{\infty} j\theta_j)$.

Finally we prove that the operator T is non-compact. The sequence $(x_n)_{n\in\mathbb{N}}$ is bounded, while from the biorthogonality of sequence $(x_n, x^*_n)_{n\in\mathbb{N}}$ it follows that for $i < j$,

$$\|Tx_{q_i} - Tx_{q_j}\| = \|\sum_{n=1}^{\infty} x^*_{q_n}(x_{q_i} - x_{q_j})e_n\| = \|e_i - e_j\| \ge \frac{1}{2K}$$

where K is the basis constant of $(e_n)_{n\in\mathbb{N}}$. Therefore T is a non-compact operator. $\square$

Proposition 3.2. There exists a strictly singular non-compact operator $S : T_0 \to T_0$.

Proof. Let X,Y denote the spaces $X = T_0 = T[(\mathcal{A}_{n_j}, \frac{1}{m_j})_{j\in\mathbb{N}}]$, $Y = T_0' = T[(\mathcal{A}_{n_{j+1}}, \frac{1}{m_j})_{j\in\mathbb{N}}]$ respectively and let K_0, K_0' be their standard norming sets. From Proposition 2.13 there exists a block sequence $(x_n^*)_{n\in\mathbb{N}}$ in $X^* = T_0^*$ which generates a c_0 spreading model. We also select a bounded block sequence $(x_n)_{n\in\mathbb{N}}$ in T_0 with $\operatorname{ran} x_n = \operatorname{ran} x_n^*$ such that $x_n^*(x_n) = 1$. The standard basis $(e_n)_{n\in\mathbb{N}}$ is a normalized Schauder basis of the space Y, while for every j and for every $\phi \in K_0'$ it holds that

$$\#\{n\in\mathbb{N}:\ |\phi(e_n)| > \frac{1}{m_j}\} \le (n_j)^2$$

(the proof of this statement follows similarly with those of Lemma 2.9 and of the claim in the proof of Lemma 4.7). Proposition 3.1 yields the existence of a strictly increasing sequence of integers $(q_n)_{n\in\mathbb{N}}$ such that the operator $T: T_0 \to T_0'$ defined by the rule

$$T(x) = \sum_{n=1}^{\infty} x_{q_n}^*(x) e_n$$

is bounded.

Since the norming set K_0 of T_0 is a subset of the norming set K_0' of T_0', the formal identity map $I: T_0' \to T_0$ defines a bounded linear operator. We show that the operator I is strictly singular. Let Y_1 be any block subspace of T_0' and let $j\in\mathbb{N}$. We may select a block sequence $(y_i)_{i=1}^{n_{j+1}}$ in Y_1 such that the sequence $(Iy_i)_{i=1}^{n_{j+1}}$ is a $(3, \frac{1}{m_{j+1}^2})$ R.I.S. in T_0 with $\|Iy_i\|_{T_0} \ge 1$, and thus $\|y_i\|_{T_0'} \ge 1$. (The technical details for the above argument and the definition of R.I.S. are similar to those of Definition 4.9, Lemmas 4.13, 4.14, 4.15 and Proposition 4.16.) From the analogue of Proposition 4.11 for the space T_0 it follows that

$$\left\| \frac{Iy_1 + Iy_2 + \cdots + Iy_{n_{j+1}}}{n_{j+1}} \right\|_{T_0} \le \frac{6}{m_{j+1}}.$$

On the other hand, selecting $f_i \in K_0'$ with $\operatorname{ran} f_i \subset \operatorname{ran} y_i$ and $f_i(y_i) \ge 1$ for $i = 1,2,\ldots,n_{j+1}$, the functional

$$f = \frac{1}{m_j}(f_1 + f_2 + \cdots + f_{n_{j+1}})$$

belongs to the norming set K_0', while its action yields that

$$\left\| \frac{y_1 + y_2 + \cdots + y_{n_{j+1}}}{n_{j+1}} \right\|_{T_0'} \ge \frac{1}{m_j}.$$

Therefore the vector $y = \frac{1}{n_{j+1}} \sum\limits_{i=1}^{n_{j+1}} y_i$ belongs to the subspace Y_1 and

$$\frac{\|Iy\|_{T_0}}{\|y\|_{T_0'}} \le \frac{\frac{6}{m_{j+1}}}{\frac{1}{m_j}} = \frac{6}{m_j}.$$

Since this procedure may be done for arbitrarily large j, it follows that the operator I is strictly singular.

We define the operator $S: T_0 \to T_0$ as the composition $S = I \circ T$. The operator S is strictly singular (as I is). It is also non-compact, since for the bounded sequence $(x_{q_n})_{n\in\mathbb{N}}$ it holds that for all $i \ne j$ we have that

$$\|S(x_{q_i}) - S(x_{q_j})\|_{T_0} = \|e_i - e_j\|_{T_0} = 1.$$

□

4. The HI space $\mathfrak{X}_d$

In this section we define the space $\mathfrak{X}_d$ and we show that it is Hereditarily Indecomposable. The unconditional frame we use in the construction of the space $\mathfrak{X}_d$ is the space T_0 we have constructed in section 2. For the definition of $\mathfrak{X}_d$ we define a Gowers-Maurey type coding function σ and we define the n_{2j-1} special sequences.

Definition 4.1. [The space $\mathfrak{X}_d$.] Let the sequences $(m_j)_{j=1}^{\infty}$, $(n_j)_{j=1}^{\infty}$ be as in Definition 2.8. The set K_d is the minimal subset of $c_{00}(\mathbb{N})$ satisfying the following conditions.

(i) $\{\pm e_k^* :\ k \in \mathbb{N}\} \subset K_d$.
(ii) K_d is symmetric (i.e. if $f \in K_d$ then $-f \in K_d$).
(iii) K_d is closed under the restriction of its elements on intervals of $\mathbb{N}$ (i.e. if $f \in K_d$ and E is an interval of $\mathbb{N}$ then $Ef \in K_d$).
(iv) For every $j \in \mathbb{N}$, K_d is closed under the $(\mathcal{A}_{n_j}, \frac{1}{m_j})$ operation.
(v) For every $j \geq 2$, K_d is closed under the $(\mathcal{A}_{n_{2j-1}}, \frac{1}{\sqrt{m_{2j-1}}})$ operation on n_{2j-1} special sequences, i.e. for every n_{2j-1} special sequence $(f_1, f_2, \ldots, f_{n_{2j-1}})$, with $f_i \in K_d$ for $1 \leq i \leq n_{2j-1}$, the functional $f = \frac{1}{\sqrt{m_{2j-1}}}(f_1 + f_2 + \cdots + f_{n_{2j-1}})$ also belongs to K_d.

The space $\mathfrak{X}_d$ is the completion of $(c_{00}(\mathbb{N}), \|\cdot\|_{K_d})$.

The above definition is not complete because we have not yet defined the n_{2j-1} special sequences.

Definition 4.2. [The coding function σ and the n_{2j-1} special sequences.] Let $\mathbb{Q}_s$ denote the set of all finite sequences $(\phi_1, \phi_2, \ldots, \phi_d)$ such that $\phi_i \in c_{00}(N)$, $\phi_i \neq 0$ with $\phi_i(n) \in \mathbb{Q}$ for all i, n and $\phi_1 < \phi_2 < \cdots < \phi_d$. We fix a pair Ω_1, Ω_2 of disjoint infinite subsets of $\mathbb{N}$. From the fact that $\mathbb{Q}_s$ is countable we are able to define an injective coding function $\sigma : \mathbb{Q}_s \to \{2j :\ j \in \Omega_2\}$ such that $m_{\sigma(\phi_1,\phi_2,\ldots,\phi_d)} > \max\{\frac{1}{|\phi_i(e_l)|} :\ l \in \operatorname{supp}\phi_i,\ i = 1, \ldots, d\} \cdot \max\operatorname{supp}\phi_d$.

Let $j \in \mathbb{N}$. A finite sequence $(f_i)_{i=1}^{n_{2j-1}}$ is said to be an n_{2j-1} special sequence provided that

(i) $(f_1, f_2, \ldots, f_{n_{2j-1}}) \in \mathbb{Q}_s$ and $f_i \in K_d$ for $i = 1, 2, \ldots, n_{2j-1}$.
(ii) The functional f_1 is the result of an $(\mathcal{A}_{n_{2k}}, \frac{1}{m_{2k}})$ operation, on a family of functionals belonging to of K_d, for some for some $k \in \Omega_1$ such that $m_{2k}^{1/2} > n_{2j-1}$ and for each $1 \leq i < n_{2j-1}$ the functional f_{i+1} is the result of an $(\mathcal{A}_{n_{\sigma(f_1,\cdots,f_i)}}, \frac{1}{m_{\sigma(f_1,\cdots,f_i)}})$ operation on a family of functionals belonging to K_d.

As we have mentioned earlier the weight $w(f)$ of a functional $f \in K_d$ is not unique. However, when we refer to an n_{2j-1} special sequence $(f_i)_{i=1}^{n_{2j-1}}$ then, for $2 \leq i \leq n_{2j-1}$, by $w(f_i)$ we shall always mean $w(f_i) = m_{\sigma(f_1,\ldots,f_{i-1})}$.

Proposition 4.3. [The tree-like property of n_{2j-1} special sequences] Let $\Phi = (\phi_i)_{i=1}^{n_{2j-1}}$, $\Psi = (\psi_i)_{i=1}^{n_{2j-1}}$ be a pair of distinct n_{2j-1} special sequences. Then

(i) For $1 \leq i < l \leq n_{2j-1}$ we have that $w(\phi_i) \neq w(\psi_l)$.
(ii) There exists $k_{\Phi,\Psi}$ such that $\phi_i = \psi_i$ for $i < k_{\Phi,\Psi}$ and $w(\phi_i) \neq w(\psi_i)$ for $i > k_{\Phi,\Psi}$.

We leave the easy proof to the reader.

Remark 4.4. We mention that, since $\sqrt{m_{2j-1}} = m_{2j-2}$ for each j, (see Definition 2.8) condition (v) in Definition 4.1 is equivalent saying that K_d is closed under the $(\mathcal{A}_{n_{2j+1}}, \frac{1}{m_{2j}})$ operation on n_{2j+1} special sequences for each j.

We call $2j+1$ special functional, every functional of the form Eh with E an interval and h the result of a $(\mathcal{A}_{n_{2j+1}}, \frac{1}{m_{2j}})$ operation on an n_{2j+1} special sequence.

Let's observe that each $f \in K_d$ is either of the form $f = \pm e_k^*$ or there exists $j \in \mathbb{N}$ such that f takes the form $f = \frac{1}{w(f)} \sum_{i=1}^{d} f_i$ with $d \leq n_{2j+1}$ and $w(f) = m_{2j+1}$ or $w(f) = m_{2j}$.

Remark 4.5. The trees of functionals $f \in K_d$ and the order $o(f)$ of such functionals are defined in a similar manner as in Definition 2.4 and Definition 2.6.

The rest of the present section is devoted to the proof of the HI property of the space $\mathfrak{X}_d$. We need to introduce the auxiliary spaces T', T'_{j_0}.

Definition 4.6. [The auxiliary spaces T', T'_{j_0}.] Let T' be mixed Tsirelson space

$$T' = T[(\mathcal{A}_{4n_i}, \frac{1}{m_i})_{i\in\mathbb{N}},\ (\mathcal{A}_{4n_{2j+1}}, \frac{1}{m_{2j}})_{j\in\mathbb{N}}]$$

and we denote by W' the standard norming set corresponding to this space. This means that W' is the minimal subset of $c_{00}(\mathbb{N})$ containing $\{\pm e_k^* :\ k \in \mathbb{N}\}$ and being closed in the $(\mathcal{A}_{4n_i}, \frac{1}{m_i})_{i\in\mathbb{N}}$ and in the $(\mathcal{A}_{4n_{2j+1}}, \frac{1}{m_{2j}})_{j\in\mathbb{N}}$ operations.

We also consider, for each $j_0 \in \mathbb{N}$, the auxiliary space

$$T'_{j_0} = T[(\mathcal{A}_{4n_i}, \frac{1}{m_i})_{i\in\mathbb{N}},\ (\mathcal{A}_{4n_{2j+1}}, \frac{1}{m_{2j}})_{j\neq j_0}]$$

and we denote by W'_{j_0} its standard norming set.

Lemma 4.7. Let $j \in \mathbb{N}$ and $f \in W'$. We have that

$$|f(\frac{1}{n_{2j}} \sum_{k=1}^{n_{2j}} e_k)| \leq \begin{cases} \frac{2}{m_i \cdot m_{2j}}, & \text{if } w(f) = m_i,\ i < 2j \\ \frac{1}{m_i}, & \text{if } w(f) = m_i,\ i \geq 2j \end{cases}$$

Proof. The case $i \geq 2j$ is obvious. For the case $i < 2j$ we need the following claim. (We shall also use the next claim later in the proofs of Proposition 5.2 and Lemma 7.1.)

Claim. If $g \in W'$ and $j \in \mathbb{N}$ then

$$\#\{k \in \mathbb{N} :\ |g(e_k)| > \frac{1}{m_{2j}}\} \leq (4n_{2j-1})^4.$$

Proof. Without loss of generality we may assume that $g(e_k) > \frac{1}{m_{2j}}$ for every $k \in \operatorname{supp} g$. Then the functional g has a tree in which appear only the operations $(\mathcal{A}_{4n_i}, \frac{1}{m_i})_{1\leq i\leq 2j-1}$ and $(\mathcal{A}_{4n_{2i+1}}, \frac{1}{m_{2i}})_{i<j}$. Then (see the proof of Lemma 2.9 and the comments before its statement) $\|g\|_q \leq 1$ where

$$\begin{aligned} q &= \max\Big(\{\log_{m_i}(4n_i) :\ 1 \leq i \leq 2j-1\} \cup \{\log_{m_{2i}}(4n_{2i+1}) :\ i < j\}\Big) \\ &= \log_{m_{2j-2}}(4n_{2j-1}). \end{aligned}$$

Hence $1 \geq \|g\|_q \geq \frac{1}{m_{2j}} \cdot \big(\#(\operatorname{supp} g)\big)^{\frac{1}{q}}$. Therefore

$$\#(\operatorname{supp} g) \leq m_{2j}^q = m_{2j-2}^{4q} = (4n_{2j-1})^4.$$

□

Let now $f \in W'$ with $w(f) = m_i$, $i < 2j$. Then the functional f takes the form $f = \frac{1}{m_i}\sum_{r=1}^{d} f_r$ with $f_1 < f_2 < \cdots < f_d$ in W' and $d \leq 4n_{2j-1}$.

We set $D_r = \{l : \ |f_r(e_l)| > \frac{1}{m_{2j}}\}$ for $r = 1, 2, \ldots, d$ and $D = \bigcup_{r=1}^{d} D_r$. From the claim above we get that $\#(D_r) \leq (4n_{2j-1})^4$ for each r, thus $\#(D) \leq (4n_{2j-1})^5$. Therefore

$$\begin{aligned}
|f(\frac{1}{n_{2j}}\sum_{k=1}^{n_{2j}} e_k)| &\leq |f_{|D}(\frac{1}{n_{2j}}\sum_{k=1}^{n_{2j}} e_k)| + |f_{|(\mathbb{N}\setminus D)}(\frac{1}{n_{2j}}\sum_{k=1}^{n_{2j}} e_k)| \\
&\leq \frac{1}{m_i}\cdot\frac{1}{n_{2j}}\cdot\#(D) + \frac{1}{m_i}\cdot\frac{1}{m_{2j}} \\
&\leq \frac{1}{m_i}(\frac{(4n_{2j-1})^5}{n_{2j}} + \frac{1}{m_{2j}}) \\
&\leq \frac{1}{m_i}(\frac{1}{m_{2j}} + \frac{1}{m_{2j}}) = \frac{2}{m_i\cdot m_{2j}}.
\end{aligned}$$

□

Lemma 4.8. Let $f \in W'_{j_0}$. Then

$$|f(\frac{1}{n_{2j_0+1}}\sum_{k=1}^{n_{2j_0+1}} e_k)| \leq \begin{cases} \frac{2}{m_{2j_0+1}m_i}\,, & \text{if } w(f) = m_i,\ i \leq 2j_0 \\ \frac{1}{m_i}\quad, & \text{if } w(f) = m_i,\ i \geq 2j_0+1 \end{cases}$$

and therefore $|f(\frac{1}{n_{2j_0+1}}\sum_{k=1}^{n_{2j_0+1}} e_k)| \leq \frac{1}{m_{2j_0+1}}$.

Proof. The estimate for $i \geq 2j_0+1$ is obvious. For the case $i \leq 2j_0$ we shall use the following claim.

Claim. For every $g \in W'_{j_0}$, we have that $\#\{k : \ |g(e_k)| > \frac{1}{m_{2j_0+1}}\} \leq (4n_{2j_0})^2$.

Proof. Let $g \in W'_{j_0}$. Without loss of generality, we may assume that $g(e_k) > \frac{1}{m_{2j_0+1}}$ for every $k \in \operatorname{supp} g$. The functional g then, has a tree in which appear only the operations $(\mathcal{A}_{4n_i}, \frac{1}{m_i})_{i\leq 2j_0}$ and $(\mathcal{A}_{4n_{2i+1}}, \frac{1}{m_{2i}})_{i<j_0}$. Then $\|g\|_q \leq 1$, where

$$q = \max\big(\{\log_{m_i}(4n_i) : \ i \leq 2j_0\} \cup \{\log_{m_{2i}}(4n_{2i+1}) : \ i < j_0\}\big) = \log_{m_{2j_0}}(4n_{2j_0}).$$

It follows that $1 \geq \|g\|_q \geq \frac{1}{m_{2j_0+1}}\#(\operatorname{supp} g)^{1/q}$ therefore

$$\#(\operatorname{supp}(g)) \leq m_{2j_0+1}^q = m_{2j_0+1}^{\log_{m_{2j_0}}(4n_{2j_0})} = m_{2j_0}^{2\log_{m_{2j_0}}(4n_{2j_0})} = (4n_{2j_0})^2.$$

□

Let $f \in W'_{j_0}$ with $w(f) = m_i$, $i \leq m_{2j_0}$. Then the functional f takes the form $f = \frac{1}{m_i}\sum_{r=1}^{d} f_r$ with $d \leq 4n_{2j_0}$. For $r = 1, \ldots, d$ we set $D_r = \{k : \ |f_r(e_k)| > \frac{1}{m_{2j_0+1}}\}$.

We also set $D = \bigcup_{r=1}^{d} D_r$. Then, using the claim, we get that $\#(D) \leq \sum_{r=1}^{d} \#(D_r) \leq d \cdot (4n_{2j_0})^2 \leq (4n_{2j_0})^3$. Therefore

$$\begin{aligned}
|f(\frac{1}{n_{2j_0+1}} \sum_{k=1}^{n_{2j_0+1}} e_k)| &\leq |f_{|D}(\frac{1}{n_{2j_0+1}} \sum_{k=1}^{n_{2j_0+1}} e_k)| + |f_{|(\mathbb{N}\setminus D)|}(\frac{1}{n_{2j_0+1}} \sum_{k=1}^{n_{2j_0+1}} e_k)| \\
&\leq \frac{1}{m_i} \cdot \frac{1}{n_{2j_0+1}} \cdot \#(D) + \frac{1}{m_i} \cdot \frac{1}{m_{2j_0+1}} \\
&\leq \frac{2}{m_i \cdot m_{2j_0+1}}.
\end{aligned}$$

The proof of the lemma is complete. □

Definition 4.9. [R.I.S.] A block sequence $(x_k)_k$ in $\mathfrak{X}_d$ is said to be a (C, ε) rapidly increasing sequence (R.I.S.), if $\|x_k\| \leq C$ for all k, and there exists a strictly increasing sequence $(j_k)_k$ of positive integers such that

(a) $\frac{1}{m_{j_1}} \leq \varepsilon$ and $\frac{1}{m_{j_{k+1}}} \cdot \# \operatorname{supp} x_k \leq \varepsilon$ for each k.
(b) For every $k = 1, 2, \ldots$ and every $f \in K_d$ with $w(f) = m_i$, $i < j_k$ we have that $|f(x_k)| \leq \frac{C}{m_i}$.

The sequence $(j_k)_k$ is called the associated sequence of the R.I.S. $(x_k)_k$.

The next proposition is the fundamental tool for providing upper bounds of the norm for certain vectors in $\mathfrak{X}_d$.

Proposition 4.10. [The basic inequality] Let $(x_k)_k$ be a (C, ε) R.I.S. in $\mathfrak{X}_d$ with associated sequence $(j_k)_k$, and let $(\lambda_k)_k$ be a sequence of scalars. Then for every $f \in K_d$ and every interval I there exists a functional

$$g \in W' = W[(\mathcal{A}_{4n_j}, \frac{1}{m_j})_{j\in\mathbb{N}}, (\mathcal{A}_{4n_{2j+1}}, \frac{1}{m_{2j}})_{j\in\mathbb{N}}]$$

with either $w(g) = w(f)$ of $g = e_r^*$ such that

$$|f(\sum_{k\in I} \lambda_k x_k)| \leq C\Big(g(\sum_{k\in I} |\lambda_k| e_k) + \varepsilon \sum_{k\in I} |\lambda_k|\Big).$$

Moreover if f is the result of an $(\mathcal{A}_{n_i}, \frac{1}{m_i})$ operation then either $g = e_r^*$ or g is the result of an $(\mathcal{A}_{4n_i}, \frac{1}{m_i})$operation.

If we additionally assume that for some $2j_0 + 1 < j_1$ we have that for every subinterval J of I and every $2j_0 + 1$ special functional f it holds that

$$|f(\sum_{k\in J} \lambda_k x_k)| \leq C\Big(\max_{k\in J} |\lambda_k| + \varepsilon \sum_{k\in J} |\lambda_k|\Big). \tag{3}$$

then we may select the functional g to be in

$$W'_{j_0} = W[(\mathcal{A}_{4n_j}, \frac{1}{m_j})_{j\in\mathbb{N}}, (\mathcal{A}_{4n_{2j+1}}, \frac{1}{m_{2j}})_{j\neq j_0}].$$

Proof. We first treat the case that for some j_0, the additional assumption (3) in the statement of the proposition is satisfied. We proceed by induction on the order $o(f)$ of the functional f.

If $o(f) = 1$, i.e. if $f = \pm e_r^*$, then we set $g = e_k^*$ for the unique $k \in I$ for which $r \in \operatorname{ran}(x_k)$ if such a k exists, otherwise we set $g = 0$.

Suppose now that the result holds for every functional in K_d with order less than q and consider $f \in K_d$ with $o(f) = q$. Then

$$f = \frac{1}{w(f)}(f_1 + f_2 + \cdots + f_d)$$

where $f_1 < f_2 < \cdots < f_d$ are in K_d with $o(f_i) < q$, and either $w(f) = m_j$ and $d \leq n_j$, or f is a $2j+1$ special functional (then $w(f) = \sqrt{m_{2j+1}} = m_{2j}$ and $d \leq n_{2j+1}$). We distinguish four cases.

Case 1. f is a $2j_0 + 1$ special functional.
We choose $k_0 \in I$ with $|\lambda_{k_0}| = \max_{k \in I} |\lambda_k|$ and we set $g = e^*_{k_0}$. Then from our assumption (3) it follows that

$$\begin{aligned} |f(\sum_{k\in I} \lambda_k x_k)| &\leq C(\max_{k\in I} |\lambda_k| + \varepsilon \sum_{k\in I} |\lambda_k|\Big) \\ &\leq C\Big(g(\sum_{k\in I} |\lambda_k| e_k) + \varepsilon \sum_{k\in I} |\lambda_k|\Big). \end{aligned}$$

Case 2. $w(f) < m_{j_k}$ for all $k \in I$ and f is not a $2j_0 + 1$ special functional.
For $i = 1, \dots, d$ we set $E_i = \operatorname{ran}(f_i)$, and

$$I_i = \{k \in I : \operatorname{ran}(x_k) \cap E_i \neq \emptyset \text{ and } \operatorname{ran}(x_k) \cap E_{i'} = \emptyset \text{ for all } i' \in I \setminus \{i\}\}.$$

We also set

$$I_0 = \big\{k \in I : \operatorname{ran}(x_k) \cap E_i \neq \emptyset \text{ for at least two } i \in \{1, \dots, \}\big\}$$

and $I' = I \setminus \bigcup_{i=0}^{d} I_i$.

We observe that $|I_0| \leq d$. For each $k \in I_0$ assumption (b) in the definition of R.I.S. yields that

$$|f(x_k)| \leq \frac{C}{w(f)}. \tag{4}$$

Observe also, that for each $i = 1, \dots, d$, I_i is a subinterval of I, hence our inductive assumption yields that there exists $g \in W'_{j_0}$ with $\operatorname{supp} g_i \subset I_i$ such that

$$|f_i(\sum_{k\in I_i} \lambda_k x_k)| \leq C\Big(g_i(\sum_{k\in I_i} |\lambda_k| e_k) + \varepsilon \sum_{k\in I_i} |\lambda_k|\Big). \tag{5}$$

The family $\{I_1, \dots, I_d\} \cup \{\{k\} : k \in I_0\}$ consists of pairwise disjoint intervals and has cardinality less than or equal to $2d$. We set

$$g = \frac{1}{w(f)}\Big(\sum_{i=1}^{d} g_i + \sum_{k\in I_0} e^*_k\Big).$$

Then $g \in W'_{j_0}$, $\operatorname{supp} g \subset I$, while from (4),(5) we get that

$$\begin{aligned}
|f(\sum_{k\in I}\lambda_k x_k)| &\leq \sum_{k\in I_0}|\lambda_k||f(x_k)| + \frac{1}{w(f)}\sum_{i=1}^{d}|f_i(\sum_{k\in I_i}\lambda_k x_k)| \\
&\leq \sum_{k\in I_0}|\lambda_k|\frac{C}{w(f)} + \frac{1}{w(f)}\sum_{i=1}^{d} C\Big(g_i(\sum_{k\in I_i}|\lambda_k|e_k) + \varepsilon\sum_{k\in I_i}|\lambda_k|\Big) \\
&\leq C\Big(g(\sum_{k\in I}|\lambda_k|e_k) + \varepsilon\sum_{k\in I}|\lambda_k|\Big).
\end{aligned}$$

Case 3. $m_{j_{k_0}} \leq w(f) < m_{j_{k_0}+1}$ for some $k_0 \in I$.
In this case, for $k \in I$ with $k < k_0$ we have that $m_{j_{k+1}} \leq m_{j_{k_0}} \leq w(f)$, hence, using assumption (a) in the definition of R.I.S. it follows that

$$|f(x_k)| \leq \frac{1}{w(f)}\|x_k\|_{\ell_1} \leq \frac{1}{m_{j_{k+1}}}\cdot C\cdot \#\operatorname{supp}(x_k) \leq C\varepsilon. \tag{6}$$

For $k \in I$ with $k > k_0$, from assumptions (a), (b) in the definition of R.I.S. we get that

$$|f(x_k)| \leq \frac{C}{w(f)} \leq \frac{C}{m_{j_1}} \leq C\varepsilon. \tag{7}$$

Thus, setting $g = e^*_{k_0}$ and using (6), (7) we get that

$$\begin{aligned}
|f(\sum_{k\in I}\lambda_k x_k)| &\leq |\lambda_{k_0}||f(x_{k_0})| + \sum_{\substack{k\in I\\ k\neq k_0}}|\lambda_k||f(x_k)| \\
&\leq |\lambda_{k_0}|C + \sum_{\substack{k\in I\\ k\neq k_0}}|\lambda_k|C\varepsilon \\
&\leq C\Big(g(\sum_{k\in I}|\lambda_k|e_k + \varepsilon\sum_{k\in I}|\lambda_k|\Big)
\end{aligned}$$

Case 4. $m_{j_{k+1}} \leq w(f)$ for all $k \in I$.
In this case, as in Case 3, we get that $|f(x_k)| \leq C\varepsilon$ for all $k \in I$ so we may set $g = 0$.

This completes the proof in the case we have made the additional assumption about j_0. When no assumption about j_0 is made, the induction is similar to the previous one, with the only difference concerning Case 2, where we include f which is a $2j_0+1$ special functional (thus Case 1 does not appear). In each inductive step the resulting functional g belongs to W'. $\square$

From Proposition 4.10 and Lemma 4.7 we conclude the following.

Proposition 4.11. Let $(x_k)_{k=1}^{n_{2j}}$ be a (C,ε) R.I.S. with $\varepsilon \leq \frac{1}{m_{2j}^2}$. Let also $f \in K_d$. Then

$$|f(\frac{1}{n_{2j}}\sum_{k=1}^{n_{2j}}x_k)| \leq \begin{cases} \frac{3C}{m_{2j}m_i}\,, & \text{if } w(f) = m_i,\ i < 2j \\ \frac{C}{m_i} + C\varepsilon\,, & \text{if } w(f) = m_i,\ i \geq 2j \end{cases}$$

In particular $\|\frac{1}{n_{2j}}\sum\limits_{k=1}^{n_{2j}}x_k\| \leq \frac{2C}{m_{2j}}$.

Definition 4.12. A vector $x \in \mathfrak{X}_d$ is said to be a $C - \ell_1^k$ average if x takes the form $x = \frac{1}{k}\sum_{i=1}^{k} x_i$, with $\|x_i\| \leq C$ for each i, $x_1 < \cdots < x_k$ and $\|x\| \geq 1$.

Lemma 4.13. Let Y be a block subspace of $\mathfrak{X}_d$ and let $k \in \mathbb{N}$. Then there exists a vector $x \in Y$ which is a $2 - \ell_1^k$ average.

For a proof we refer to [12] Lemma II.22.

Lemma 4.14. If x is a $C - \ell_1^k$ average, $d \leq k$ and $E_1 < \cdots < E_d$ is a sequence of intervals then $\sum_{i=1}^{d} \|E_i x\| \leq C(1 + \frac{2d}{k})$. In particular if x is a $C - \ell_1^{n_{2j}}$ average then for every $f \in K_d$ with $w(f) = m_i$, $i < 2j$ we have that $|f(x)| \leq \frac{1}{m_i} C(1 + \frac{2n_{2j-1}}{n_{2j}}) \leq \frac{3C}{2}\frac{1}{w(f)}$.

For a proof we refer to [12] Lemma II.23. The next lemma is a direct consequence of Lemma 4.14.

Lemma 4.15. Let $(x_k)_{k\in\mathbb{N}}$ be a block sequence in $\mathfrak{X}_d$ such that each x_k is a $C - \ell_1^{n_{2l_k}}$ average, where $(l_k)_{k\in\mathbb{N}}$ is a strictly increasing sequence of integers, and let $\varepsilon > 0$. Then there exists a subsequence of $(x_k)_{k\in\mathbb{N}}$ which is a $(\frac{3C}{2}, \varepsilon)$ R.I.S.

Proposition 4.16. [Existence of R.I.S.] For every $\varepsilon > 0$ and every block subspace Z of $\mathfrak{X}_d$ there exists a $(3, \varepsilon)$ R.I.S. $(x_k)_{k\in\mathbb{N}}$ in Z with $\|x_k\| \geq 1$.

Proof. It follows from Lemma 4.13 and Lemma 4.15. $\square$

Definition 4.17. [Exact pairs.] A pair (x, ϕ) with $x \in \mathfrak{X}_d$ and $\phi \in K_d$ is said to be a $(C, 2j)$ exact pair (where $C \geq 1$, $j \in \mathbb{N}$) if the following conditions are satisfied:

(i) $1 \leq \|x\| \leq C$, for every $\psi \in K_d$ with $w(\psi) < m_{2j}$, we have that $|\psi(x)| \leq \frac{3C}{w(\psi)}$, while for $\psi \in K_d$ with $w(\psi) > m_{2j}$, $|\psi(x)| \leq \frac{C}{m_{2j}^2}$.

(ii) $\phi \in K_d$ with $w(\phi) = m_{2j}$.

(iii) $\phi(x) = 1$ and $\operatorname{ran} x = \operatorname{ran} \phi$.

Proposition 4.18. Let $j \in \mathbb{N}$. Then for every block subspace Z of $\mathfrak{X}_d$, there exists a $(6, 2j)$ exact pair (x, ϕ) with $x \in Z$.

Proof. From Proposition 4.16 there exists $(x_k)_{k=1}^{n_{2j}}$ a $(3, \varepsilon)$-R.I.S. in Z with $\varepsilon \leq \frac{1}{2m_{2j}^3}$ and $\|x_k\| \geq 1$. Choose $x_k^* \in K_d$ with $x_k^*(x_k) \geq 1$ and $\operatorname{ran} x_k^* = \operatorname{ran} x_k$. Then Proposition 4.11 yields that for some θ with $\frac{1}{6} \leq \theta \leq 1$,

$$(\theta \frac{m_{2j}}{n_{2j}} \sum_{k=1}^{n_{2j}} x_k, \frac{1}{m_{2j}} \sum_{k=1}^{n_{2j}} x_k^*)$$

is a $(6, 2j)$ exact pair. $\square$

Definition 4.19. [Dependent sequences.] A double sequence $(x_k, x_k^*)_{k=1}^{n_{2j+1}}$ with $x_k \in \mathfrak{X}_d$ and $x_k^* \in K_d$ is said to be a $(C, 2j+1)$ dependent sequence if there exists a sequence $(2j_k)_{k=1}^{n_{2j+1}}$ of even integers such that the following conditions are fulfilled:

(i) $(x_k^*)_{k=1}^{n_{2j+1}}$ is an n_{2j+1} special sequence with $w(x_k^*) = m_{2j_k}$ for all $1 \leq k \leq n_{2j+1}$.

(ii) Each (x_k, x_k^*) is a $(C, 2j_k)$ exact pair.

Remark 4.20. It follows easily, that if $(x_k, x_k^*)_{k=1}^{n_{2j+1}}$ is a $(C, 2j+1)$ dependent sequence then the sequence $(x_k)_{k=1}^{n_{2j+1}}$ is a $(3C, \varepsilon)$ R.I.S. where $\varepsilon = \frac{1}{n_{2j+1}^2}$.

Proposition 4.21. Let $j \in \mathbb{N}$. Then for every pair of block subspaces Z, W of $\mathfrak{X}_d$ there exists a $(6, 2j+1)$ dependent sequence $(x_k, x_k^*)_{k=1}^{n_{2j+1}}$ with $x_{2k-1} \in Z$ and $x_{2k} \in W$ for all k.

Proof. It follows easily from an inductive application of Proposition 4.18. $\square$

We need the next lemma in order to apply Proposition 4.10 with the additional assumption.

Lemma 4.22. Let $(x_k, x_k^*)_{k=1}^{n_{2j+1}}$ be a $(C, 2j+1)$ dependent sequence. Then for every $2j+1$ special functional f and every subinterval I of $\{1, 2, \ldots, n_{2j+1}\}$ we have that $|f(\sum\limits_{k \in I} (-1)^{k+1} x_k)| \leq C$.

Proof. The functional f takes the form

$$f = \frac{1}{m_{2j}}(Ex_t^* + x_{t+1}^* + \cdots + x_{r-1}^* + f_r + f_{r+1} + \cdots + f_d)$$

where $(x_1^*, x_2^*, \ldots, x_{r-1}^*, f_r, f_{r+1}, \ldots, f_{n_{2j+1}})$ is an n_{2j+1} special sequence with $w(f_r) = w(x_r^*)$, $f_r \neq x_r^*$, E is an interval of the form $E = [m, \max\operatorname{supp} x_t^*]$ and $d \leq n_{2j+1}$.

Using the definitions of dependent sequences and exact pairs we obtain the following.
For $k < t$ we have that $f(x_k) = 0$.
For $k = t$, $|f(x_t)| = \frac{1}{m_{2j}} |Ex_t^*(x_t)| \leq \frac{1}{m_{2j}} \cdot \|x_t\| \leq \frac{C}{m_{2j}}$.
For $t < k < r$, we get that $f(x_k) = \frac{1}{m_{2j}} x_k^*(x_k) = \frac{1}{m_{2j}}$.
For the case $k = r$ we shall say later.
Let k with $r < k \leq n_{2j+1}$. For $i \leq r-1$ we have that $\operatorname{ran}(x_i^*) \cap \operatorname{ran} x_k = \emptyset$ thus $x_i^*(x_k) = 0$. Also, the injectivity of the coding function σ yields that $w(f_i) \neq m_{2j_k} = w(x_k^*)$ for $r \leq i \leq d$. Setting

$$J_k^- = \{i : \ w(f_i) < m_{2j_k}\} \quad \text{and} \quad J_k^+ = \{i : \ w(f_i) > m_{2j_k}\}$$

we get that

$$\begin{aligned}
|f(x_k)| &\leq \frac{1}{m_{2j}}\Big(\sum_{i \in J_k^-} |f_i(x_k)| + \sum_{i \in J_k^+} |f_i(x_k)|\Big) \\
&\leq \frac{1}{m_{2j}}\Big(\sum_{i \in J_k^-} \frac{3C}{w(f_i)} + \sum_{i \in J_k^+} \frac{C}{m_{2j_k}^2}\Big) \\
&\leq \frac{C}{m_{2j}}\Big(\frac{4}{w(x_1^*)} + n_{2j+1} \cdot \frac{1}{m_{2j_1}^2}\Big) \\
&\leq \frac{C}{m_{2j}}\Big(\frac{4}{n_{2j+1}^2} + n_{2j+1} \cdot \frac{1}{n_{2j+1}^4}\Big) \leq \frac{5C}{m_{2j}} \cdot \frac{1}{n_{2j+1}^2}.
\end{aligned}$$

For $k = r$ using similar arguments it follows that $|f(x_r)| \leq \frac{2C}{m_{2j}}$.

We set $I_1 = I \cap \{t\}$, $I_2 = I \cap \{t+1, \dots, r-1\}$, $I_3 = I \cap \{r\}$, $I_4 = I \cap \{r+1, \dots, n_{2j+1}\}$ and we conclude that

$$\begin{aligned} |f(\sum_{k\in I}(-1)^{k+1}x_k)| &\leq \sum_{k\in I_1}|f(x_k)| + |f(\sum_{k\in I_2}(-1)^{k+1}x_k)| \\ &\quad + \sum_{k\in I_3}|f(x_k)| + \sum_{k\in I_4}|f(x_k)| \\ &\leq \frac{C}{m_{2j}} + \frac{1}{m_{2j}} + \frac{2C}{m_{2j}} + n_{2j+1}\cdot\frac{5C}{m_{2j}}\cdot\frac{1}{n_{2j+1}^2} \\ &\leq C. \end{aligned}$$

□

Proposition 4.23. Let $(x_k, x_k^*)_{k=1}^{n_{2j+1}}$ be a $(C, 2j+1)$ dependent sequence. Then

$$\|\frac{1}{n_{2j+1}}\sum_{k=1}^{n_{2j+1}}(-1)^{k+1}x_k\| \leq \frac{4C}{m_{2j+1}}.$$

Proof. The sequence $(x_k)_{k=1}^{n_{2j+1}}$ is a $(3C, \varepsilon)$ R.I.S. for $\varepsilon = \frac{1}{n_{2j+1}^2}$ (Remark 4.20). It follows from Lemma 4.22 that the additional assumption of Proposition 4.10 concerning the number $j_0 = j$ and the sequence $(\frac{(-1)^{k+1}}{n_{2j+1}})_{k=1}^{n_{2j+1}}$ is fulfilled. Thus applying Proposition 4.10 and Lemma 4.8 we get that for every $f \in K_d$ there exists $g \in W_j'$ such that

$$\begin{aligned} |f(\frac{1}{n_{2j+1}}\sum_{k=1}^{n_{2j+1}}(-1)^{k+1}x_k)| &\leq 3C\big(g(\frac{1}{n_{2j+1}}\sum_{k=1}^{n_{2j+1}}e_k) + \frac{1}{n_{2j+1}^2}\big)\big) \\ &\leq 3C(\frac{1}{m_{2j+1}} + \frac{1}{n_{2j+1}^2}) \\ &\leq \frac{4C}{m_{2j+1}}. \end{aligned}$$

This completes the proof of the proposition. □

Theorem 4.24. The space $\mathfrak{X}_d$ is a reflexive HI space.

Proof. The Schauder basis $(e_n)_{n\in\mathbb{N}}$ of the space $\mathfrak{X}_d$ is boundedly complete and shrinking (this follows by similars arguments with the corresponding result in [23]). Therefore $\mathfrak{X}_d$ is a reflexive space.

Let Z, W be a pair of infinite dimensional subspaces of $\mathfrak{X}_d$. We shall show that for every $\varepsilon > 0$ there exist $z \in Z$, $w \in W$ with $\|z-w\| < \varepsilon\|z+w\|$. It is easy to check that this yields the HI property of $\mathfrak{X}_d$. From the well known gliding hump argument we may assume that Z, W are block subspaces. Then for $j \in \mathbb{N}$, using Proposition 4.21, we select $(x_k, x_k^*)_{k=1}^{n_{2j+1}}$ a $(6, 2j+1)$ dependent sequence with $x_{2k-1} \in Z$ and $x_{2k} \in W$ for all k. From Proposition 4.23 we get that

$$\|\frac{1}{n_{2j+1}}\sum_{k=1}^{n_{2j+1}}(-1)^{k+1}x_k\| \leq \frac{24}{m_{2j+1}}.$$

On the other hand, since $(x_k^*)_{k=1}^{n_{2j+1}}$ is an n_{2j+1} special sequence, the functional $f = \frac{1}{\sqrt{m_{2j+1}}}\sum_{k=1}^{n_{2j+1}} x_k^*$ belongs to the norming set K_d, while the action of f on the vector $\frac{1}{n_{2j+1}}\sum\limits_{k=1}^{n_{2j+1}} x_k$ yields that

$$\|\frac{1}{n_{2j+1}}\sum_{k=1}^{n_{2j+1}} x_k\| \geq \frac{1}{\sqrt{m_{2j+1}}}.$$

Thus setting $z=\sum\limits_{k=1}^{n_{2j+1}/2} x_{2k-1}$ and $w=\sum\limits_{k=1}^{n_{2j+1}/2} x_{2k}$ we get that $z\in Z,\ w\in W$ and $\|z-w\|\leq \frac{24}{\sqrt{m_{2j+1}}}\|z+w\|$ which for sufficiently large j yields the desired result.

Therefore the space $\mathfrak{X}_d$ is HI. $\square$

5. A class of bounded diagonal operators on $\mathfrak{X}_d$

In this section we present the construction of a class of bounded diagonal operators on the space $\mathfrak{X}_d$. These operators are of the form $\sum\limits_k \lambda_k D_{j_k}$ where $\{j_k:\ k\in\mathbb{N}\}$ is a lacunary set and $(\lambda_k)_{k\in\mathbb{N}}$ is any bounded sequence of real numbers. Each D_{j_k} is of the form $D_{j_k}(x)=\frac{1}{m_{j_k}}\sum\limits_{i=1}^{p_k} I_i^{j_k}x$. We pass to the details of the construction.

Let $\{I_i^j:\ 1\leq i\leq p_j,\ j=1,2,\ldots\}$ be any family of intervals of $\mathbb{N}$ such that, for every j

$$I_1^j<I_2^j<\cdots<I_{p_j}^j<I_1^{j+1}.$$

For each $j\in\mathbb{N}$, we define the diagonal operator $D_j:\mathfrak{X}_d\to\mathfrak{X}_d$ by the rule

$$D_j(x)=\frac{1}{m_j}\sum_{i=1}^{p_j} I_i^j x.$$

We also define

$$\alpha_j(x)=\frac{1}{m_j}\sum_{i=1}^{p_j}\|I_i^j x\|$$

and we observe that for every $j\in\mathbb{N}$ and $x\in\mathfrak{X}_d$ we have that

$$\|D_j x\|\leq \alpha_j(x)\leq \|x\|.$$

Indeed, the left inequality is obvious while, in order to prove the right one, for $i=1,\ldots,p_j$, we select $\phi_i\in K_d$ such that $\text{supp}(\phi_i)\subset I_i^j$ and $\phi_i(x)=\|I_i^j x\|$. Then, $\phi=\frac{1}{m_j}\sum\limits_{i=1}^{p_j}\phi_i\in K_d$, thus

$$\alpha_j(x)=\frac{1}{m_j}\sum_{i=1}^{p_j}\|I_i^j x\|=\phi(x)\leq\|x\|.$$

Lemma 5.1. Let $L\subset\mathbb{N}$ with $\#L\leq \min L$. Then, for every $x\in\mathfrak{X}_d$, we have that

$$\sum_{j\in L}\alpha_j(x)\leq\|x\|.$$

Proof. Let $L=\{j_1,j_2,\ldots,j_s\}$ with $s\leq j_1<j_2<\cdots<j_s$. For every $k=1,\ldots,s$ and $i=1,\ldots,p_{j_k}$ we choose $\phi_i^k\in K_d$ such that $\text{supp}(\phi_i^k)\subset I_i^{j_k}$ and $\phi_i^k(x)=\|I_i^{j_k}x\|$. Then, for every $k=1,\ldots,s$, we have that $\phi^k=\frac{1}{m_{j_k}}\sum\limits_{i=1}^{p_{j_k}}\phi_i^k\in K_d$ and

$\phi^k(x) = \alpha_{j_k}(x)$. Moreover, as in the proof of Proposition 2.13, the functional $f = \sum_{k=1}^{s} \phi^k$ takes the form

$$f = \frac{1}{m_{j_1}} \sum_{l=1}^{n_{j_1}} \psi_l$$

with $(\psi_l)_{l=1}^{n_{j_1}}$ being successive members of K_d, hence $f \in K_d$. It follows that

$$\sum_{k=1}^{s} \alpha_{j_k}(x) = f(x) \leq \|x\|.$$

□

Proposition 5.2. Let $M = \{j_k : \ j \in \mathbb{N}\}$ be a subset of $\mathbb{N}$ such that for every k the following conditions are satisfied:

(i) $m_{j_{k+1}} \geq 2^{k+1} \cdot n_{j_k+1}$.
(ii) $m_{j_{k+1}} \geq 2^k \cdot \max I^{j_k}_{p_{j_k}}$.
(iii) $j_k > n_{2k}$.

Then for every $(\lambda_k)_{k\in\mathbb{N}} \in \ell_\infty(\mathbb{N})$, the operator $\sum_k \lambda_k D_{j_k}$ is bounded and strictly singular with

$$\|\sum_k \lambda_k D_{j_k}\| \leq C_0 \cdot \sup_k |\lambda_k|$$

where $C_0 = 3 + \sum_{i=1}^{\infty} \frac{i+1}{m_{2i}}$.

We divide the proof of Proposition 5.2 in several steps. The main step is done in the following proposition.

Proposition 5.3. For every $f \in K_d$ and every interval I there exists $g \in W'$ (recall that W' is the norming set of the space $T' = T[(\mathcal{A}_{4n_i}, \frac{1}{m_i})_{i\in\mathbb{N}},\ (\mathcal{A}_{4n_{2j+1}}, \frac{1}{m_{2j}})_{j\in\mathbb{N}}]$, see Definition 4.6) having nonnegative coordinates, with $\operatorname{supp} g \subset I$, such that for every $x \in \mathfrak{X}_d$ is holds that

$$|f(D_{j_k}x)| \leq \alpha_{j_k}(x) g(e_k) + \frac{1}{2^k}\|x\|$$

for all $k \in I$ with the potential exception for $k \in \{k_0, k_0+1\}$ where $k_0 + 1 < \operatorname{supp} g$.

For the proof, we need the following Lemma.

Lemma 5.4. Let $k \in \mathbb{N}$, $\phi \in K_d$ and $x \in \mathfrak{X}_d$.

(i) If $w(\phi) \leq m_{j_{k-1}}$ then

$$|\phi(D_{j_k}(x))| = \left| \phi\left(\frac{1}{m_{j_k}} \sum_{i=1}^{p_{j_k}} I_i^{j_k} x \right) \right| \leq \frac{1}{w(\phi)} \alpha_{j_k}(x) + \frac{1}{2^k}\|x\|.$$

(ii) If $w(\phi) \geq m_{j_{k+1}}$ then $|\phi(D_{j_k}(x))| \leq \frac{1}{2^k}\|x\|$.

Proof. (i) Let $\phi \in K_d$ with $w(\phi) \leq m_{j_{k-1}}$. Then $\phi = \frac{1}{w(\phi)} \sum_{l=1}^{d} \phi_l$ where, for some $j \in \mathbb{N}$, either $w(\phi) = m_j$ or $w(\phi) = \sqrt{m_j}$ and $d \leq n_j$. Since $w(\phi) \leq m_{j_{k-1}}$, in either case we get that $d \leq n_{j_{k-1}+1}$.

For $l=1,\dots,d$, we set

$$R_l=\big\{i:\ \mathrm{ran}(\phi_l)\cap I_i^{j_k}\neq\emptyset \text{ and } \mathrm{ran}(\phi_{l'})\cap I_i^{j_k}=\emptyset \text{ for } l'\neq l\big\}.$$

We also set

$$A=\big\{i\in\{1,\dots,p_{j_k}\}:\ \mathrm{ran}(\phi_l)\cap I_i^{j_k}\neq\emptyset \text{ for at least two } l\big\}.$$

It is easy to see that $\#A\le d\le n_{j_{k-1}+1}$. For every $x\in\mathfrak{X}_d$ we get that

$$\begin{aligned}|\phi(D_{j_k}(x))| &= \Big|\frac{1}{m_{j_k}}\sum_{i\in A}\phi(I_i^{j_k}x)+\frac{1}{w(\phi)}\frac{1}{m_{j_k}}\sum_{l=1}^{d}\phi_l\big(\sum_{i\in R_l}I_i^{j_k}x\big)\Big|\\ &\le \frac{1}{m_{j_k}}\sum_{i\in A}\|I_i^{j_k}x\|+\frac{1}{w(\phi)}\frac{1}{m_{j_k}}\sum_{i=1}^{p_{j_k}}\|I_i^{j_k}x\|\\ &\le \frac{1}{m_{j_k}}n_{j_{k-1}+1}\|x\|+\frac{1}{w(\phi)}\alpha_{j_k}(x).\end{aligned}$$

From property (i) of the sequence $(j_k)_{k=1}^{\infty}$, we get that

$$|\phi(D_{j_k}(x))|\le\frac{1}{2^k}\|x\|+\frac{1}{w(\phi)}\alpha_{j_k}(x).$$

(ii) Let now $\phi\in K_d$ with $w(\phi)\ge m_{j_{k+1}}$ and $x\in\mathfrak{X}_d$. We have that

$$\begin{aligned}\big|\phi\big(D_{j_k}(x)\big)\big| &= \left|\phi\left(\frac{1}{m_{j_k}}\sum_{i=1}^{p_{j_k}}I_i^{j_k}x\right)\right|\le\frac{1}{m_{j_k}}\frac{1}{w(\phi)}\left\|\sum_{i=1}^{p_{j_k}}I_i^{j_k}x\right\|_{\ell_1}\\ &\le \frac{1}{w(\phi)}\cdot\max I_{p_{j_k}}^{j_k}\cdot\|x\|_\infty\le\frac{1}{m_{j_{k+1}}}\cdot\max I_{p_{j_k}}^{j_k}\cdot\|x\|\\ &\le \frac{1}{2^k}\|x\|\end{aligned}$$

where the last inequality follows from property (ii) of the sequence $(j_k)_{k=1}^{\infty}$. $\square$

Proof of Proposition 5.3. For each $k=1,2,\dots$ let I^k be the minimal interval containing $\bigcup_{i=1}^{p_{j_k}}I_i$. We proceed to the proof by induction on the order $o(f)$ of the functional f.

If $o(f)=1$, i.e. if $f=\pm e_r^*$, then, if $r\in I^k$ for some $k\in I$ we set $g=e_k^*$, otherwise we set $g=0$. Suppose now that the conclusion holds for every functional in K_d having order less than q and consider $f\in K_d$ with $o(f)=q$. Then $f=\frac{1}{w(f)}(f_1+f_2+\cdots+f_d)$ with $o(f_i)<q$ for each i, while either $w(f)=m_j$ and $d\le n_j$ for some j, or f is a $2j+1$ special functional, in which case $w(f)=\sqrt{m_{2j+1}}=m_{2j}$ and $d\le n_{2j+1}$.

For $i=1,\dots,d$ we set

$$I_i=\big\{k\in I:\ \mathrm{ran}(f_i)\cap I^k\neq\emptyset \text{ and } \mathrm{ran}(f_{i'})\cap I^k=\emptyset \text{ for } i'\in I\setminus\{i\}\big\}.$$

We also set

$$I_0=\big\{k\in I:\ \mathrm{ran}(f_i)\cap I^k\neq\emptyset \text{ for at least two } i\big\}$$

and we observe that $\#I_0\le d$.

Let now $k_0\in\mathbb{N}$ be such that $m_{j_{k_0}}\le w(f)<m_{j_{k_0+1}}$ (the modifications in the rest of the proof are obvious if no such k_0 exists, i.e. if $w(f)<m_{j_1}$). For $k<k_0$, Lemma 5.4 (ii) yields that $|f(D_{j_k}(x))|\le\frac{1}{2^k}\|x\|$ for every $x\in\mathfrak{X}_d$, while for

$k > k_0 + 1$, Lemma 5.4 (i) yields that $|f(D_{j_k}(x))| \leq \frac{1}{w(f)}\alpha_{j_k}(x) + \frac{1}{2^k}\|x\|$ for every $x \in \mathfrak{X}_d$.

For each $i = 1, \ldots, d$ from our inductive assumption there exists $g_i \in W'$ with $\operatorname{supp} g_i \subset I_i$ such that

$$|f(D_{j_k}(x))| \leq \frac{1}{w(f)}\alpha_{j_k}(x) + \frac{1}{2^k}\|x\|$$

for all $k \in I_i$, with the potential exception for $k \in \{k_i, k_i+1\}$ where $k_i+1 < \operatorname{supp} g_i$. For the rest of the proof suppose that $k_i, k_i + 1 \in I_i$ are indeed exceptions to the above inequality.

We set

$$g = \frac{1}{w(f)}\Big(\sum_{i=1}^{d}(e^*_{k_i} + e^*_{k_i+1} + g_i) + \sum_{k \in I_0} e^*_k\Big)$$

and $g = [k_0 + 2, +\infty)g'$. The family $\{e^*_{k_i},\ e^*_{k_i+1},\ g_i,\ i = 1, \ldots, d\} \cup \{e^*_k :\ k \in I_0\}$ consists of successive functionals belonging to W', while its cardinality does not exceed $4d$. Thus the functional g' belongs to W' hence the same holds for the functional g. We have to check that the functional g satisfies the conclusion of the proposition.

Let $x \in \mathfrak{X}_d$. For $k < k_0$, as we have observed earlier, we have that $|f(D_{j_k}(x))| \leq \frac{1}{2^k}\|x\|$. The numbers $k_0, k_0 + 1$, if belong to I, are the potential exceptions to the required inequality; observe also that $k_0+1 < \operatorname{supp} g$. Let now $k \in I$ with $k > k_0+1$. We distinguish four cases.

Case 1. $k \in \{k_i, k_i + 1\} \subset I_i$ for some $i \in \{1, \ldots, d\}$.
Then

$$|f(D_{j_k}(x))| = \frac{1}{w(f)}|f_i(D_{j_k}(x))| \leq \frac{1}{w(f)}\|D_{j_k}(x)\| \leq \frac{1}{w(f)}\alpha_{j_k}(x) = \alpha_{j_k}(x)g(e_k).$$

Case 2. $k \in I_i \setminus \{k_i, k_i + 1\}$ for some $i \in \{1, \ldots, d\}$.
Then

$$\begin{aligned}|f(D_{j_k}(x))| &= \frac{1}{w(f)}|f_i(D_{j_k}(x))| \leq \frac{1}{w(f)}\big(\alpha_{j_k}(x)g_i(e_k) + \frac{1}{2^k}\|x\|\big)\\ &\leq \alpha_{j_k}(x)g(e_k) + \frac{1}{2^k}\|x\|.\end{aligned}$$

Case 3. $k \in I_0$.
Then, since also $k > k_0 + 1$ we get that

$$|f(D_{j_k}(x))| \leq \frac{1}{w(f)}\alpha_{j_k}(x) + \frac{1}{2^k}\|x\| = \alpha_{j_k}(x)g(e_k) + \frac{1}{2^k}\|x\|.$$

Case 4. $k \in I \setminus \bigcup_{i=0}^{d} I_i$.
Then $|f(D_{j_k}(x))| = 0$.

The proof of the proposition is complete. $\square$

Lemma 5.5. Let $g \in W'$ and $x \in \mathfrak{X}_d$. Then

$$\sum_{k=1}^{\infty}\alpha_{j_k}(x)|g(e_k)| \leq C_1\|x\|$$

where $C_1 = \sum_{i=1}^{\infty}\frac{i+1}{m_{2i}}$.

Proof. We set

$$F_1 = \{k : \frac{1}{m_2} < |g(e_k)|\}$$

and for $i = 2, 3, \dots$ we set

$$F_i = \{k : \frac{1}{m_{2i}} < |g(e_k)| \le \frac{1}{m_{2i-2}}\}.$$

Since $m_1 = m_2 = 2$, if $F_1 \neq \emptyset$ then $g = \pm e_r^*$ and the conclusion trivially follows (since $C_1 \ge 1$). Suppose now that $F_1 = \emptyset$. From the claim in the proof of Lemma 4.7 we get that $\#F_i \le (4n_{2i-1})^4 \le n_{2i}$ for each $i = 2, 3, \dots$. We set

$$L_i = \{k \in F_i : \ n_{2i} < j_k\} \ mboxand \ G_i = F_i \setminus L_i = \{k \in F_i : \ j_k \le n_{2i}\}.$$

Since

$$\#\{j_k : k \in L_i\} \le \#L_i \le \#F_i \le n_{2i} < \min\{j_k : k \in L_i\},$$

Lemma 5.1 yields that

$$\sum_{k \in L_i} \alpha_{j_k}(x) \le \|x\|.$$

On the other hand, by Property (iii) of the sequence $(j_k)_{k=1}^{\infty}$, we have $n_{2i} < j_i$, and hence, $G_i \subset \{1, \dots, i-1\}$. Thus, for $i \ge 2$,

$$\sum_{k \in F_i} \alpha_{j_k}(x) = \sum_{k \in G_i} \alpha_{j_k}(x) + \sum_{k \in L_i} \alpha_{j_k}(x) \le (i-1)\|x\| + \|x\| = i\|x\|.$$

We conclude that

$$\begin{aligned}
\sum_{k=1}^{\infty} \alpha_{j_k}(x)|g(e_k)| &= \sum_{i=2}^{\infty} \sum_{k \in F_i} \alpha_{j_k}(x)|g(e_k)| \\
&\le \sum_{i=2}^{\infty} \frac{1}{m_{2i-2}} \big(\sum_{k \in F_i} \alpha_{j_k}(x) \big) \\
&\le \big(\sum_{i=2}^{\infty} \frac{i}{m_{2i-2}} \big) \|x\| = C_1 \|x\|.
\end{aligned}$$

□

Proof of Proposition 5.2. Firstly we shall show the bound of the norm of the operator $D = \sum_k \lambda_k D_{j_k}$. Let $x \in \mathfrak{X}_d$. We shall show that for every $f \in K_d$, it holds that

$$|f(\sum_k \lambda_k D_{j_k}(x))| \le C_0 \cdot \sup_k |\lambda_k| \cdot \|x\|.$$

Let $f \in K_d$. From Proposition 5.3 there exists $g \in W'$ having nonnegative coordinates and $k_0 \in \mathbb{N}$ such the

$$|f(D_{j_k}(x))| \le \alpha_{j_k}(x) g(e_k) + \frac{1}{2^k}\|x\|$$

for all $k \notin \{k_0, k_0+1\}$. Therefore

$$\begin{aligned}
|f(\sum_k \lambda_k D_{j_k}(x))| &\leq \sum_k |\lambda_k| \cdot |f(D_{j_k}(x))| \leq \sup_k |\lambda_k| \cdot \sum_k |f(D_{j_k}(x))| \\
&\leq \sup_k |\lambda_k| \cdot \Bigg(|f(D_{j_{k_0}}(x))| + |f(D_{j_{k_0+1}}(x))| \\
&\qquad + \sum_{k \notin \{k_0, k_0+1\}} \big(\alpha_{j_k}(x) g(e_k) + \frac{1}{2^k}\|x\| \big) \Bigg) \\
&\leq \sup_k |\lambda_k| \cdot \Bigg(\|D_{j_{k_0}}(x)\| + \|D_{j_{k_0+1}}(x)\| + \sum_{k=1}^{\infty} \frac{1}{2^k}\|x\| \\
&\qquad + \sum_{k=1}^{\infty} \alpha_{j_k}(x) g(e_k) \Bigg) \\
&\leq \sup_k |\lambda_k| \cdot \Bigg(3\|x\| + \sum_{k=1}^{\infty} \alpha_{j_k}(x) g(e_k) \Bigg).
\end{aligned}$$

From Lemma 5.5 we get that

$$\sum_{k=1}^{\infty} \alpha_{j_k}(x) g(e_k) \leq C_1 \|x\|,$$

where $C_1 = \sum_{i=1}^{\infty} \frac{i+1}{m_{2i}}$. Thus the operator $D = \sum_k \lambda_k D_{j_k}$ is bounded with $\|D\| \leq C_0 \cdot \sup_k |\lambda_k|$ where $C_0 = 3 + C_1$.

The fact that the operator $D : \mathfrak{X}_d \to \mathfrak{X}_d$ is strictly singular follows from the fact that $\lim_n D(e_n) = 0$ in conjunction to the HI property of $\mathfrak{X}_d$ (see Proposition 1.2 of [13]). □

6. The structure of the space $\mathcal{L}_{\text{diag}}(\mathfrak{X}_d)$

In this section we define the space J_{T_0}, which is the Jamesification of the space T_0 studied in section 2. We state the finitely block representability of J_{T_0} in $\mathfrak{X}_d$ (the proof of this result is presented in the next section) and apply it in order to study the structure of the space $\mathcal{L}_{\text{diag}}(\mathfrak{X}_d)$ of diagonal operators on $\mathfrak{X}_d$. We start with the definition of the space J_{T_0}.

Definition 6.1. The space J_{T_0} is defined to be the space

$$J_{T_0} = T\big[G, (\mathcal{A}_{n_j}, \frac{1}{m_j})_{n \in \mathbb{N}}\big]$$

where $G = \{\pm \chi_I : \ I \text{ finite interval of } \mathbb{N}\}$. This means that J_{T_0} is the completion of $(c_{00}(\mathbb{N}), \|\cdot\|_{D_0})$ where D_0 is the minimal subset of $c_{00}(\mathbb{N})$ such that:

(i) The set G is a subset of D_0.
(ii) The set D_0 is closed in the $(\mathcal{A}_{n_j}, \frac{1}{m_j})$ operation for every $j \in \mathbb{N}$.

Remark 6.2. An alternative description of the space J_{T_0} is the following. Let $(t_n)_{n \in \mathbb{N}}$ be the standard Hamel basis of $c_{00}(\mathbb{N})$. The norm $\|\cdot\|_{J_{T_0}}$ is defined as follows: For every $x=\sum\limits_{n=1}^{\infty}x(n)t_n\in c_{00}(\mathbb{N})$ we set

$$\|x\|_{J_{T_0}}=\sup\big\{\|\sum_{k=1}^{l}\big(\sum_{n\in I_k}x(n)\big)e_k\|_{T_0},\ l\in\mathbb{N},\ I_1<I_2<\ldots<I_l \text{ intervals of } \mathbb{N}\big\}.$$

The space J_{T_0} is the completion of $(c_{00}(\mathbb{N}),\|\cdot\|_{J_{T_0}})$.

Proposition 6.3. For the space J_{T_0} the following hold.

(i) The sequence $(t_n)_n$ is a normalized bimonotone Schauder basis of the space J_{T_0}.

(ii) For every $j\in\mathbb{N}$, we have the following estimates:

$$\|\frac{1}{2p_j}\sum_{k=1}^{p_j}t_{2k-1}\|_{J_{T_0}}=\frac{1}{2}$$

$$\|\frac{1}{2p_j}\sum_{k=1}^{2p_j}(-1)^{k+1}t_k\|_{J_{T_0}}=\|\frac{1}{2p_j}\sum_{k=1}^{2p_j}e_k\|_{T_0}\le\frac{4}{m_j}.$$

In particular the basis $(t_n)_{n\in\mathbb{N}}$ is not unconditional.

Proof. The proof that $(t_n)_{n\in\mathbb{N}}$ is a normalized bimonotone Schauder basis is standard. We set $x=\frac{1}{2p_j}\sum\limits_{k=1}^{p_j}t_{2k-1}$. The inequality $\|x\|_{J_{T_0}}\le\frac{1}{2}$ is obvious while from the action of the functional $f=\chi_I\in D_0$, where $I=\{1,2,\ldots,2p_j-1\}$, on the vector x we obtain that $\|x\|_{J_{T_0}}\ge\frac{1}{2}$.

Setting $l=2p_j$ and $I_k=\{k\}$ for $1\le k\le l$ we get $\|\sum\limits_{k=1}^{2p_j}(-1)^{k+1}t_k\|_{J_{T_0}}\ge\|\sum\limits_{k=1}^{2p_j}(-1)^{k+1}e_k\|_{T_0}=\|\sum\limits_{k=1}^{2p_j}e_k\|_{T_0}$ where the inequality follows from Remark 6.2, while the equality is a consequence of the 1-unconditionality of the basis $(e_k)_{k\in\mathbb{N}}$ of T_0 (Remarks 2.5 and 2.7).

Let's explain now the inequality $\|\sum\limits_{k=1}^{2p_j}(-1)^{k+1}t_k\|_{J_{T_0}}\le\|\sum\limits_{k=1}^{2p_j}e_k\|_{T_0}$. We observe that for every interval I of $\mathbb{N}$ the quantity $\sum\limits_{k\in I}(-1)^{k+1}$ is either equal to -1 or to 0 or to 1. Thus the inequality follows from Remarks 6.2 and 2.5.

Finally the inequality $\|\frac{1}{2p_j}\sum\limits_{k=1}^{2p_j}e_k\|_{T_0}\le\frac{4}{m_j}$ follows from Lemma 2.10. $\square$

Theorem 6.4. There exists a positive constant c such that the basis $(t_n)_{n\in\mathbb{N}}$ of J_{T_0} is c - finitely representable in every block subspace of $\mathfrak{X}_d$. This means that, for every block subspace Z of $\mathfrak{X}_d$ and every $N\in\mathbb{N}$, there exists a finite block sequence $(z_k)_{k=1}^N$ in Z such that, for every choice of scalars $(\mu_k)_{k=1}^N$, we have that

$$\|\sum_{k=1}^{N}\mu_k t_k\|_{J_{T_0}}\le\|\sum_{k=1}^{N}\mu_k z_k\|_{\mathfrak{X}_d}\le c\cdot\|\sum_{k=1}^{N}\mu_k t_k\|_{J_{T_0}}\ .$$

We shall give the proof of Theorem 6.4 in the next section. Let us note that, since the basis $(t_n)_{n\in\mathbb{N}}$ of J_{T_0} is not unconditional, Theorem 6.4 implies in particular that the space $\mathfrak{X}_d$ does not contain any unconditional basic sequence. Of course, in Theorem 4.24, we have already proved the stronger result that the space $\mathfrak{X}_d$ is Hereditarily Indecomposable.

From Theorem 6.4 and Proposition 6.3 we immediately get the following.

Corollary 6.5. Let Z be any block subspace of $\mathfrak{X}_d$ and let $j \in \mathbb{N}$. Then there exists a finite block sequence $(y_k)_{k=1}^{2p_j}$ in Z such that

$$\|\frac{1}{2p_j}\sum_{k=1}^{p_j} y_{2k-1}\| \geq \frac{1}{2} \qquad \text{and} \qquad \|\frac{1}{2p_j}\sum_{k=1}^{2p_j}(-1)^{k+1}y_k\| \leq \frac{4c}{m_j}$$

Theorem 6.6. There exist bounded strictly singular non-compact diagonal operators on the space $\mathfrak{X}_d$. Especially, given any infinite dimensional subspace Z of $\mathfrak{X}_d$ there exists a bounded strictly singular diagonal operator on $\mathfrak{X}_d$ such that its restriction on Z is a non-compact operator.

Moreover the space $\mathcal{L}_{\text{diag}}(\mathfrak{X}_d)$ of all bounded diagonal operators on the space $\mathfrak{X}_d$ contains an isomorphic copy of $\ell_\infty(\mathbb{N})$.

Proof. By standard perturbation arguments and passing to a subspace we may assume that Z is a block subspace of $\mathfrak{X}_d$. We inductively construct vectors $(y_k^j)_{k=1}^{2p_j}$, $j = 1, 2, \ldots$ in Z, satisfying the conclusion of Corollary 6.5 and moreover $y_{2p_j}^j < y_1^{j+1}$ for each j.

For $j = 1, 2 \ldots$ and $1 \leq i \leq p_j$ we set $I_i^j = \text{ran}(y_{2i-1}^j)$ and we define the diagonal operator $D_j : \mathfrak{X}_d \to \mathfrak{X}_d$ by the rule $D_j(x) = \frac{1}{m_j}\sum\limits_{i=1}^{p_j} I_i^j x$. We also consider for $j = 1, 2, \ldots$ the vector $x_j = \frac{m_j}{2p_j}\sum\limits_{k=1}^{2p_j}(-1)^{k+1}y_k^j$ which belongs to Z. Then $\|x_j\| \leq 4c$, $\|D_j x_j\| \geq \frac{1}{2}$ while $D_l x_j = 0$ for $l \neq j$.

Let now $M = \{j_k : \ k \in \mathbb{N}\}$ be any subset of $\mathbb{N}$ satisfying conditions (i), (ii), (iii) in the statement of Proposition 5.2. Then, from Proposition 5.2, the diagonal operator $D = \sum\limits_{k=1}^{\infty} D_{j_k}$ is bounded and strictly singular. The restriction of D on Z is non-compact, since the block sequence $(x_{j_k})_{k\in\mathbb{N}}$ is bounded, while the sequence $(Dx_{j_k})_{k\in\mathbb{N}}$ does not have any convergent subsequence.

For $M = \{j_k : \ k \in \mathbb{N}\}$ as above, Proposition 5.2 yields that for every $(\lambda_k)_{k\in\mathbb{N}} \in \ell_\infty(\mathbb{N})$, the diagonal operator $\sum\limits_{k=1}^{\infty}\lambda_k D_{j_k}$ is bounded with $\|\sum\limits_{k=1}^{\infty}\lambda_k D_{j_k}\| \leq C_0 \cdot \sup\limits_k |\lambda_k|$. On the other hand the action of the operator $\sum\limits_{k=1}^{\infty}\lambda_k D_{j_k}$ to the vector x_{j_m} yields that $\|\sum\limits_{k=1}^{\infty}\lambda_k D_{j_k}\| \geq \frac{|\lambda_m|\cdot\|D_{j_m}(x_{j_m})\|}{\|x_{j_m}\|} \geq \frac{1}{8c}\cdot|\lambda_m|$ for each m. Hence

$$\frac{1}{8c}\cdot\sup_k|\lambda_k| \leq \|\sum_{k=1}^{\infty}\lambda_k D_{j_k}\| \leq C_0\cdot\sup_k|\lambda_k|.$$

The proof of the theorem is complete. □

7. The finite block representability of J_{T_0} in $\mathfrak{X}_d$

The content of this section is the proof of Theorem 6.4. Let $N \in \mathbb{N}$ and let Z be any block subspace of $\mathfrak{X}_d$. We first choose $j \geq 2$ with $2p_j \geq N$ and $i > j$ such that $m_{2i-1} > 38p_j$. Then we select $(x_r, \phi_r)_{r=1}^{n_{2i+1}}$ a $(6, 2i+1)$ dependent sequence with $x_r \in Z$ and $\min\operatorname{supp} x_1 > m_{2i+1}$ (this is done with an inductive application of Proposition 4.18). The fact that $(\phi_r)_{r=1}^{n_{2i+1}}$ is a special sequence yields that the functional

$$\Phi = \frac{1}{m_{2i}}(\phi_1 + \phi_2 + \cdots + \phi_{n_{2i+1}}).$$

is a $2i+1$ special functional and thus belong to the norming set K_d.

We set $M = \frac{n_{2i+1}}{2p_j}$ and observe that $M \geq (4n_{2i})^2$. For $1 \leq k \leq 2p_j$ we set

$$y_k = \frac{m_{2i}}{M} \sum_{r=(k-1)M+1}^{kM} x_r.$$

We also consider the functionals

$$y_k^* = \frac{1}{m_{2i}} \sum_{r=(k-1)M+1}^{kM} \phi_r$$

for $1 \leq k \leq 2p_j$, and we notice that $y_k^* \in K_d$ (since each y_k^* is the restriction of Φ on some interval) with $\operatorname{ran} y_k = \operatorname{ran} y_k^*$ and $\|y_k\| \geq y_k^*(y_k) = 1$. Observe also, that for every subinterval I of $\{1, 2, \ldots, 2p_j\}$, the functionals $\pm \sum\limits_{k \in I} y_k^*$ also belong to K_d.

Our aim is prove that for every choice of scalars $(\mu_k)_{k=1}^{2p_j}$ we have that

$$\|\sum_{k=1}^{2p_j} \mu_k t_k\|_{J_{T_0}} \leq \|\sum_{k=1}^{2p_j} \mu_k y_k\| \leq 150 \cdot \|\sum_{k=1}^{2p_j} \mu_k t_k\|_{J_{T_0}}. \tag{8}$$

This will finish the proof of Theorem 6.4 for $c = 150$. We begin with the proof of the left side inequality of (8) which is the easy one.

Proof of the left side inequality of (8)**.** It is enough to prove that for every choice of scalars $(\mu_k)_{k=1}^{2p_j}$ and every $g \in D_0$ (recall that D_0 is the norming set of the space J_{T_0}; see Definition 6.1) there exists $f \in K_d$ such that $g\big(\sum\limits_{k=1}^{2p_j} \mu_k t_k\big) = f\big(\sum\limits_{k=1}^{2p_j} \mu_k y_k\big)$.

Let $g \in D_0$. We may assume that $\operatorname{supp} g \subset \{1, 2, \ldots 2p_j\}$. Let $(g_a)_{a \in \mathcal{A}}$ be a tree of the functional g. We shall build functionals $(f_a)_{a \in \mathcal{A}}$ in K_d such that $g_a\big(\sum\limits_{k=1}^{2p_j} \mu_k t_k\big) = f_a\big(\sum\limits_{k=1}^{2p_j} \mu_k y_k\big)$ for each $a \in \mathcal{A}$. Then the functional $f = f_0$ (where $0 \in \mathcal{A}$ is the root of the tree $\mathcal{A}$) satisfies the desired property.

For $a \in \mathcal{A}$ which is maximal the functional g_a is of the form $g_a = \varepsilon \chi_I$ where $\varepsilon \in \{-1, 1\}$ and I is a subinterval of $\{1, 2, \ldots 2p_j\}$. We set $f_a = \varepsilon \sum\limits_{k \in I} y_k^*$ and the desired equality holds since $y_k^*(y_k) = 1$ for each k. Let now $a \in \mathcal{A}$ be non maximal and suppose that the functionals $(f_\beta)_{\beta \in S_a}$ have been defined. The functional g_a has an expression $g_a = \frac{1}{m_q} \sum\limits_{\beta \in S_a} g_\beta$ with $\# S_a \leq n_q$, for some $q \in \mathbb{N}$. We set $f_a = \frac{1}{m_q} \sum\limits_{\beta \in S_a} f_\beta$. Then $f_a \in K_d$ while the required equality is obvious. The inductive construction is complete. $\square$

Before passing to the proof of the right side inequality of (8) we need some preliminary lemmas.

Lemma 7.1. Consider the vector $x = \frac{1}{M}\sum_{l=1}^{M} e_l$ in the auxiliary space T' (recall the the auxiliary space T' and its norming set W' have been defined in Definition 4.6). Then

(i) If either $f \in W'$ with $w(f) \geq m_{2i+1}$ or f is the result of an $(\mathcal{A}_{4n_{2i+1}}, \frac{1}{m_{2i}})$ operation then

$$|f(x)| \leq \frac{1}{w(f)}.$$

(ii) If either $f \in W'$ with $w(f) < m_{2i}$ or f is the result of an $(\mathcal{A}_{4n_{2i}}, \frac{1}{m_{2i}})$ operation then

$$|f(x)| \leq \frac{2}{w(f) \cdot m_{2i}}.$$

Proof. Part (i) is obvious. In order to prove part (ii) consider $f \in W'$ such that either $w(f) < m_{2i}$ or f is the result of an $(\mathcal{A}_{4n_{2i}}, \frac{1}{m_{2i}})$ operation. In either case the functional f takes the form $f = \frac{1}{w(f)}\sum_{k=1}^{d} f_k$ with $f_1 < f_2 < \cdots < f_d$ in W' and $d \leq 4n_{2i}$. We set $D_k = \{l : \ |f_k(e_l)| > \frac{1}{m_{2i}}\}$ for $k = 1, 2, \ldots, d$ and $D = \bigcup_{k=1}^{d} D_k$. From the claim in the proof of Lemma 4.7 we get that $\#(D_k) \leq (4n_{2i-1})^4$ for each k, thus $\#(D) \leq 4n_{2i} \cdot (4n_{2i-1})^4$.

Taking into account that $M \geq (4n_{2i})^2 \geq 4n_{2i} \cdot (4n_{2i-1})^4 \cdot m_{2i}$ we deduce that

$$\begin{aligned}
|f(x)| &\leq |f_{|D}(x)| + |f_{|(\mathbb{N}\setminus D)}(x)| \\
&\leq \frac{1}{w(f)} \cdot \frac{1}{M} \cdot \#(D) + \frac{1}{w(f)} \cdot \frac{1}{m_{2i}} \\
&\leq \frac{1}{w(f)}\Big(\frac{4n_{2i} \cdot (4n_{2i-1})^4}{M} + \frac{1}{m_{2i}}\Big) \\
&\leq \frac{2}{w(f) \cdot m_{2i}}.
\end{aligned}$$

□

Lemma 7.2. For $1 \leq k \leq 2p_j$ we have the following.

(i) If either $f \in K_d$ with $w(f) < m_{2i}$ or f is the result of an $(\mathcal{A}_{n_{2i}}, \frac{1}{m_{2i}})$ operation then

$$|f(y_k)| \leq \frac{54}{w(f)}.$$

(ii) If either $f \in K_d$ with $w(f) \geq m_{2i+1}$ or f is a $2i+1$ special functional (i.e. $f = Eh$ where h is the result of a $(\mathcal{A}_{n_{2i+1}}, \frac{1}{m_{2i}})$ operation on an n_{2i+1} special sequence) then

$$|f(y_k)| \leq \frac{18m_{2i}}{w(f)} + \frac{36m_{2i}}{M} \leq \frac{19}{m_{2i}}.$$

In particular $\|y_k\| \leq 36$.

Proof. From Remark 4.20 it follows that the sequence $(x_r)_{r\in\mathbb{N}}$ (and thus every subsequence) is an $(18, \frac{1}{n_{2i+1}^2})$ R.I.S. The result follows from Proposition 4.10 and Lemma 7.1. □

Proof of the left side inequality of (8)**.** Let $f \in K_d$. We fix a tree $(f_a)_{a\in\mathcal{A}}$ of the functional f. We set

$$\mathcal{B}' = \{a \in \mathcal{A} : \ f_a \text{ is a } 2i+1 \text{ special functional}\}.$$

Let $\beta \in \mathcal{B}'$. Then the functional f_β takes the form

$$f_\beta = \varepsilon_\beta \frac{1}{m_{2i}} E(\phi_1 + \cdots \phi_{l_0} + \psi_{l_0+1} + \cdots + \psi_{n_{2i+1}})$$

where $\varepsilon_\beta \in \{-1, 1\}$, E is an interval of $\mathbb{N}$ and $(\phi_1, \dots, \phi_{l_0}, \psi_{l_0+1}, \dots, \psi_{n_{2i+1}})$ is an n_{2i+1} special sequence with $\psi_{l_0+1} \neq \phi_{l_0+1}$. For β and f_β as above, we set

$$I_\beta = \big\{k \in \{1, 2, \dots, 2p_j\} : \ \operatorname{supp} y_k \subset \operatorname{ran} E(\phi_1 + \cdots \phi_{l_0})\big\}.$$

Let

$$\mathcal{B} = \{\beta \in \mathcal{B}' : \ I_\beta \neq \emptyset\}.$$

We notice that

(i) For every $\beta \in \mathcal{B}$, the set I_β is a subinterval of $\{1, 2, \dots, 2p_j\}$.
(ii) For $\beta_1, \beta_2 \in \mathcal{B}$ with $\beta_1 \neq \beta_2$ we have that $I_{\beta_1} \cap I_{\beta_2} = \emptyset$. In particular $\sum\limits_{\beta\in\mathcal{B}} \#(I_\beta) \leq 2p_j$.
(iii) For every $\beta \in \mathcal{B}$ we have that $f_\beta(\sum\limits_{k\in I_\beta} \mu_k y_k) = \varepsilon_\beta \sum\limits_{k\in I_\beta} \mu_k$.

We set

$$F = \bigcup_{\beta\in\mathcal{B}} I_\beta.$$

Claim 1. We have $\quad |f(\sum\limits_{k\in F} \mu_k y_k)| \leq 3 \cdot \| \sum\limits_{k=1}^{2p_j} \mu_k t_k \|_{J_{T_0}}$.

Proof of Claim 1. We partition the set $\mathcal{B}$ into two subsets as follows:

$$\begin{aligned} \mathcal{B}_1 &= \{\gamma \in \mathcal{B} : \ \text{there exists } \beta \in \mathcal{B} \text{ with } \beta \prec \gamma\} \\ \mathcal{B}_2 &= \{\gamma \in \mathcal{B} : \ \beta \notin \mathcal{B} \text{ for every } \beta \prec \gamma\}. \end{aligned}$$

We shall first estimate $|f(\sum\limits_{\gamma\in\mathcal{B}_1} \sum\limits_{k\in I_\gamma} \mu_k y_k)|$. Let $\gamma \in \mathcal{B}_1$ and consider $\beta \in \mathcal{B}$ with $\beta \prec \gamma$. The functional f_β is, as we have mentioned before, of the form

$$f_\beta = \varepsilon_\beta \frac{1}{m_{2i}} E(\phi_1 + \cdots \phi_{l_0} + \psi_{l_0+1} + \cdots + \psi_{n_{2i+1}})$$

with $\phi_{l_0+1} \neq \psi_{l_0+1}$. Then $\operatorname{supp} f_\gamma \subset \operatorname{supp} \psi_l$ for some $l \geq l_0 + 1$. Since ψ_l is not a special functional we obtain that $f_\gamma \neq \psi_l$. Thus

$$|\psi_l(\sum_{k\in I_\gamma} \mu_k y_k)| \leq \frac{1}{w(\psi_l)} |f_\gamma(\sum_{k\in I_\gamma} \mu_k y_k)|.$$

From the definition of special functionals we get that $w(\psi_l) > w(\phi_1) > n_{2i+1}^2$. We also have that $|f_\gamma(\sum\limits_{k\in I_\gamma} \mu_k y_k)| = |\sum\limits_{k\in I_\gamma} \mu_k| \leq \max\limits_k |\mu_k| \cdot \#(I_\gamma)$. Thus

$$|f(\sum_{k\in I_\gamma} \mu_k y_k)| \leq |\psi_l(\sum_{k\in I_\gamma} \mu_k y_k)| \leq \frac{1}{n_{2i+1}^2} \cdot \max_k |\mu_k| \cdot \#(I_\gamma).$$

We conclude that

$$\begin{aligned}|f(\sum_{\gamma\in\mathcal{B}_1}\sum_{k\in I_\gamma}\mu_k y_k)| &\leq \sum_{\gamma\in\mathcal{B}_1}|f(\sum_{k\in I_\gamma}\mu_k y_k)| \leq \sum_{\gamma\in\mathcal{B}_1}\frac{1}{n_{2i+1}^2}\cdot\max_k|\mu_k|\cdot\#(I_\gamma)\\ &\leq \max_k|\mu_k|\cdot\frac{2p_j}{n_{2i+1}^2}\leq\max_k|\mu_k|\leq\|\sum_{k=1}^{2p_j}\mu_k t_k\|_{J_{T_0}}.\end{aligned}$$

Our next estimate concerns $|f(\sum_{\gamma\in\mathcal{B}_2}\sum_{k\in I_\gamma}\mu_k y_k)|$. From the definition of $\mathcal{B}_2$, its elements are incomparable nodes of the tree $\mathcal{A}$. We consider the minimal complete subtree $\mathcal{A}'$ of $\mathcal{A}$ containing $\mathcal{B}_2$, i.e.

$$\mathcal{A}'=\{a\in\mathcal{A}:\ \text{ there exists } \beta\in\mathcal{B}_2 \text{ with } a\preceq\beta\}.$$

For every $a\in\mathcal{A}'$ we set

$$R_a=\bigcup_{\beta\in\mathcal{B}_2,\ \beta\succeq a} I_\beta.$$

As follows from the definition of the sets I_β, for every non maximal $a\in\mathcal{A}'$, the sets $(R_\beta)_{\beta\in S_a\cap\mathcal{A}'}$ are pairwise disjoint.

For every $a\in\mathcal{A}'$ we shall construct functionals $g_a,h_a\in c_{00}(\mathbb{N})$ such that the following conditions are satisfied:

(i) $\operatorname{supp} g_a\subset R_a$ and $\operatorname{supp} h_a\subset R_a$.
(ii) $g_a\in D_0$ (the norming set D_0 of the space J_{T_0} has been defined in Definition 6.1) and $\|h_a\|_\infty\leq\frac{1}{m_{2i+1}}$.
(iii) $g_a(\sum_{k\in R_a}\mu_k t_k)\geq 0$ and $h_a(\sum_{k\in R_a}\mu_k t_k)\geq 0$.
(iv) $|f_a(\sum_{k\in R_a}\mu_k y_k)|\leq(g_a+h_a)(\sum_{k\in R_a}\mu_k t_k)$.

The construction is inductive starting of course with the maximal elements of $\mathcal{A}'$, i.e. with the elements of $\mathcal{B}_2$.

$1\stackrel{st}{=}$ **inductive step**

Let $\beta\in\mathcal{B}_2$. Then f_β is a $2i+1$ special functional, $R_\beta=I_\beta$ and $|f_\beta(\sum_{k\in R_\beta}\mu_k y_k)|=$ $|\sum_{k\in R_\beta}\mu_k|$. We set $\varepsilon=\operatorname{sgn}(\sum_{k\in R_\beta}\mu_k)$, $g_\beta=\varepsilon\cdot\chi_{I_\beta}$ and $h_a=0$. It is clear that our requirements about g_β,h_β are satisfied.

General inductive step

Let $a\in\mathcal{A}'$, $a\notin\mathcal{B}_2$ and assume that for every $\gamma\in S_a\cap\mathcal{A}'$ the functionals g_γ,h_γ have been defined satisfying the inductive assumptions. We distinguish three cases.

Case 1. f_a is not a special functional.

Let $f_a=\frac{1}{m_p}\sum_{\gamma\in S_a}f_\gamma$ with $\#S_a\leq n_p$. We set

$$g_a=\frac{1}{m_p}\sum_{\gamma\in S_a\cap\mathcal{A}'}g_\beta\qquad\text{and}\qquad h_a=\frac{1}{m_p}\sum_{\gamma\in S_a\cap\mathcal{A}'}h_\beta.$$

Conditions (i), (ii), (iii) are obviously satisfied, while, since $R_a = \bigcup_{\gamma\in S_a\cap\mathcal{A}'} R_\gamma$, we get that

$$\begin{aligned}
|f_a(\sum_{k\in R_a}\mu_k y_k)| &= |\frac{1}{m_p}\sum_{\gamma\in S_a\cap\mathcal{A}'} f_\gamma(\sum_{k\in R_a}\mu_k y_k)| \\
&\le \frac{1}{m_p}\sum_{\gamma\in S_a\cap\mathcal{A}'} |f_\gamma(\sum_{k\in R_\gamma}\mu_k y_k)| \\
&\le \frac{1}{m_p}\sum_{\gamma\in S_a\cap\mathcal{A}'} (g_\gamma+h_\gamma)(\sum_{k\in R_\gamma}\mu_k t_k) \\
&= (g_a+h_a)(\sum_{k\in R_a}\mu_k t_k).
\end{aligned}$$

Case 2. f_a is a $2q+1$ special functional for some $q \ge i$.
Then $f_a = \varepsilon_a \frac{1}{m_{2q}} E(\phi_1 + \cdots \phi_{l_0} + \psi_{l_0+1} + \cdots + \psi_{n_{2q+1}})$, with $\phi_{l_0+1} \ne \psi_{l_0+1}$ (functionals of the form ϕ_r in the expression above may appear only if $q = i$; if $q > i$ then $l_0 = 0$). If $q > i$ then $a \notin \mathcal{B}'$, hence it has no sense to talk about I_a. In the case $q = i$ from the definition of the set $\mathcal{B}_2$ we get that $I_a = \emptyset$. Similarly to the proof concerning $\mathcal{B}_1$, we obtain that for every $\beta \in \mathcal{B}_2$ with $a \prec \beta$ there exists $l \ge l_0 + 1$ such that $\operatorname{supp} f_\beta \subset \operatorname{supp}\psi_l$ and

$$|f_a(\sum_{k\in I_\beta}\mu_k y_k)| \le \frac{1}{n_{2q+1}^2}\cdot \max_{k\in I_\beta}|\mu_k|\cdot \#(I_\beta).$$

Therefore

$$\begin{aligned}
|f_a(\sum_{k\in R_a}\mu_k y_k)| &\le \sum_{\beta\in\mathcal{B}_2,\ \beta\succ a} |f_a(\sum_{k\in I_\beta}\mu_k y_k)| \le \frac{1}{n_{2q+1}^2}\cdot\max_{k\in R_a}|\mu_k|\cdot \sum_{\beta\in\mathcal{B}_2,\ \beta\succ a}\#(I_\beta) \\
&\le \frac{2p_j}{n_{2q+1}^2}\cdot\max_{k\in R_a}|\mu_k| \le \frac{1}{n_{2q+1}}\cdot\max_{k\in R_a}|\mu_k|.
\end{aligned}$$

We select $k_a \in R_a$ such that $|\mu_{k_a}| = \max_{k\in R_a}|\mu_k|$ and we set

$$g_a = 0 \qquad\text{and}\qquad h_a = \operatorname{sgn}(\mu_{k_a})\cdot\frac{1}{n_{2q+1}}\cdot t_{k_a}^*.$$

Case 3. f_a is a $2q+1$ special functional for some $q < i$.
Then f_a takes the form $f_a = \varepsilon_a\frac{1}{m_{2q}}E(f_{\gamma_1} + \cdots + f_{\gamma_d})$ with $d \le n_{2q+1}$. Similarly to the proof concerning $\beta \in \mathcal{B}_1$, for every $\beta \in \mathcal{B}_2$ with $a \prec \beta$ there exists s such that $\operatorname{supp} f_\beta \subset \operatorname{supp} f_{\gamma_s}$, while

$$|f_{\gamma_s}(\sum_{k\in I_\beta}\mu_k y_k)| \le \frac{1}{w(f_{\gamma_s})}\cdot|f_\beta(\sum_{k\in I_\beta}\mu_k y_k)|.$$

Let s_0 be such that $w(f_{\gamma_{s_0}}) < m_{2i+1} < w(f_{\gamma_{s_0+1}})$. From the definition of the special sequences and the coding function σ, we get that

$$\#\left(\bigcup_{s=1}^{s_0-1}\operatorname{ran} f_{\gamma_s}\right) \le \max\operatorname{supp} f_{\gamma_{s_0-1}} < w(f_{\gamma_{s_0}}) < m_{2i+1}.$$

Since for each k we have that $\#\operatorname{supp} y_k \ge M > m_{2i+1}$, it follows that for every $s < s_0$ there is no $\beta \in \mathcal{B}_2$ such that $\operatorname{supp} f_\beta \subset \operatorname{supp} f_{\gamma_s}$.

If $s > s_0$ and $\beta \in \mathcal{B}_2$ are such that $\operatorname{supp} f_\beta \subset \operatorname{supp} f_{\gamma_s}$ then

$$|f_{\gamma_s}(\sum_{k\in I_\beta} \mu_k y_k)| \leq \frac{1}{w(f_{\gamma_s})}|f_\beta(\sum_{k\in I_\beta} \mu_k y_k)| \leq \frac{1}{m_{2i+2}} \cdot \max_{k\in I_\beta} |\mu_k| \cdot \#(I_\beta).$$

We select $k_a \in \bigcup_{s>s_0} R_{\gamma_s}$ such that $|\mu_{k_a}| = \max\{|\mu_k| : \ k \in \bigcup_{s>s_0} R_{\gamma_s}\}$.

If there is no $\beta \in \mathcal{B}_2$ such that $\gamma_{s_0} \prec \beta$ then we set

$$g_a = 0 \qquad \text{and} \qquad h_a = \operatorname{sgn}(\mu_{k_a}) \cdot \frac{1}{m_{2i+1}} \cdot t^*_{k_a}.$$

If there exists $\beta \in \mathcal{B}_2$ such that $\gamma_{s_0} \prec \beta$ then the functionals g_{s_0} and h_{s_0} have been defined in the previous inductive step. We set

$$g_a = g_{s_0} \qquad \text{and} \qquad h_a = h_{s_0} + \operatorname{sgn}(\mu_{k_a}) \cdot \frac{1}{m_{2i+1}} \cdot t^*_{k_a}.$$

Conditions (i), (ii), (iii) are easily established; we shall show condition (iv). We assume that there exists $\beta \in \mathcal{B}_2$ such that $\gamma_{s_0} \prec \beta$ (the modifications are obvious is no such β exists).

$$\begin{aligned}
|f_a(\sum_{k\in R_a} \mu_k y_k)| &\leq |f_{\gamma_{s_0}}(\sum_{k\in R_{\gamma_{s_0}}} \mu_k y_k)| + \sum_{s>s_0} |f_{\gamma_s}(\sum_{k\in R_{\gamma_s}} \mu_k y_k)| \\
&\leq (g_{s_0} + h_{s_0})(\sum_{k\in R_{\gamma_{s_0}}} \mu_k t_k) \\
&\qquad + \frac{1}{m_{2i+2}} \cdot \max_{k\in \bigcup_{s>s_0} R_{\gamma_s}} |\mu_k| \cdot \sum_{s>s_0} \#(R_{\gamma_s}) \\
&\leq g_{s_0}(\sum_{k\in R_{\gamma_{s_0}}} \mu_k t_k) + h_{s_0}(\sum_{k\in R_{\gamma_{s_0}}} \mu_k t_k) + \frac{1}{m^2_{2i+1}} \cdot |\mu_{k_a}| \cdot 2p_j \\
&\leq g_{s_0}(\sum_{k\in R_a} \mu_k t_k) + h_{s_0}(\sum_{k\in R_a} \mu_k t_k) \\
&\qquad + \operatorname{sgn}(\mu_{k_a}) \cdot \frac{1}{m_{2i+1}} \cdot e^*_{k_a}(\sum_{k\in R_a} \mu_k t_k) \\
&= (g_a + h_a)(\sum_{k\in R_a} \mu_k t_k).
\end{aligned}$$

The inductive construction is complete.

For the the functionals g_0, h_0 corresponding to the root $0 \in \mathcal{A}$ of the tree $\mathcal{A}$, noticing that $R_0 = \bigcup_{\beta\in\mathcal{B}_2} I_\beta$, we get that

$$\begin{aligned}
|f(\sum_{\beta\in\mathcal{B}_2}\sum_{k\in I_\beta}\mu_k y_k)| &= |f(\sum_{k\in R_0}\mu_k y_k)| \leq (g_0+h_0)(\sum_{k\in R_0}\mu_k t_k)\\
&\leq g_0(\sum_{k\in R_0}\mu_k t_k) + \frac{1}{m_{2i+1}}\cdot\max_{k\in R_0}|\mu_k|\cdot\#(R_0)\\
&\leq g_0(\sum_{k=1}^{2p_j}\mu_k t_k) + \frac{2p_j}{m_{2i+1}}\cdot\max_{1\leq k\leq 2p_j}|\mu_k|\\
&\leq \|\sum_{k=1}^{2p_j}\mu_k t_k\|_{J_{T_0}} + \max_{1\leq k\leq 2p_j}|\mu_k|\\
&\leq 2\cdot\|\sum_{k=1}^{2p_j}\mu_k t_k\|_{J_{T_0}}.
\end{aligned}$$

Therefore we get that

$$\begin{aligned}
|f(\sum_{k\in F}\mu_k y_k)| &\leq |f(\sum_{\gamma\in\mathcal{B}_1}\sum_{k\in I_\gamma}\mu_k y_k)| + |f(\sum_{\beta\in\mathcal{B}_2}\sum_{k\in I_\beta}\mu_k y_k)|\\
&\leq \|\sum_{k=1}^{2p_j}\mu_k t_k\|_{J_{T_0}} + 2\cdot\|\sum_{k=1}^{2p_j}\mu_k t_k\|_{J_{T_0}} = 3\cdot\|\sum_{k=1}^{2p_j}\mu_k t_k\|_{J_{T_0}}
\end{aligned}$$

and this finishes the proof of Claim 1. $\square$

Next we shall estimate $|f(\sum_{k\notin F}\mu_k y_k)|$. We clearly may restrict our intention to $k \in D$ where

$$D = \{k \in \{1,2,\ldots,2p_j\} : \ k \notin F \text{ and } \operatorname{supp} f \cap \operatorname{supp} y_k \neq \emptyset\}.$$

In order to estimate $f(\sum_{k\in D}\mu_k y_k)$ we shall split the vector y_k, for each $k \in D$, into two parts, the initial part y_k' and the final part y_k''. The way of the split depends on the specific analysis $(f_a)_{a\in\mathcal{A}}$ of the functional f that we have fixed.

Definition 7.3. For $k \in D$ and $a \in \mathcal{A}$ we say that f_a covers y_k if

$$\operatorname{supp}(f_a)\cap\operatorname{supp}(y_k) = \operatorname{supp}(f)\cap\operatorname{supp}(y_k).$$

Next we introduce some notation which will be used in the rest of the proof.

Notation 7.4. We correspond to each y_k, for $k \in D$, two vectors y_k', y_k'' defined as follows.

Case 1. $\#\Big(\operatorname{supp}(f)\cap\operatorname{supp}(y_k)\Big) = 1.$

Then there exists a unique maximal node $a_k \in \mathcal{A}$ such $f_{a_k} = e^*_{l_k}$ covers y_k. In this case we set $y_k' = y_k$ and $y_k'' = 0$.

Case 2. $\#\Big(\operatorname{supp}(f)\cap\operatorname{supp}(y_k)\Big) \geq 2.$

Then there exists a unique node $a_k \in \mathcal{A}$ such that f_{a_k} covers y_k but for every

$\beta \in S_{a_k}$, f_β does not cover y_k. Let

$$\{\beta \in S_a : \ \operatorname{supp}(f_\beta) \cap \operatorname{supp}(y_k) \neq \emptyset\} = \{\beta_1, \beta_2, \ldots, \beta_d\}$$

with $f_{\beta_1} < f_{\beta_2} < \cdots < f_{\beta_d}$. We set

$$y_k' = y_k|_{[1, \max \operatorname{supp} f_{\beta_1}]} \qquad \text{and} \qquad y_k'' = y_k - y_k'.$$

Remark 7.5. The estimates given in Lemma 7.2 for the vectors y_k, $1 \leq k \leq 2p_j$, remain valid if we replace, for each $k \in D$, the vector y_k by either the vector y_k' or by the vector y_k''.

The analogue of Definition 7.3, concerning the vectors y_k', y_k'' is the following.

Definition 7.6. For $k \in D$ and $a \in \mathcal{A}$ we say that f_a covers y_k' if $\operatorname{supp}(f_a) \cap \operatorname{supp}(y_k') = \operatorname{supp}(f) \cap \operatorname{supp}(y_k')$ while we say that f_a covers y_k'' if $\operatorname{supp}(f_a) \cap \operatorname{supp}(y_k'') = \operatorname{supp}(f) \cap \operatorname{supp}(y_k'')$.

The property of the sequences $(y_k')_{k\in D}$ and $(y_k'')_{k \in D}$ which will play a key role in our proof is described in the following remark.

Remark 7.7. (i) Suppose that $k \in D$ and $a \in \mathcal{A}$ is a non maximal node such that f_a covers y_k' but for every $\beta \in S_a$, f_β does not cover y_k'. Then there exists a node $\beta_k \in S_a$ (not necessarily unique) such that

$$\operatorname{supp}(f_{\beta_k}) \cap \operatorname{supp}(y_k') \neq \emptyset$$

and

$$\operatorname{supp}(f_{\beta_k}) \cap \operatorname{supp}(y_l') = \emptyset \quad \text{for all } l \in D \text{ with } l \neq k.$$

(ii) The statement of (i) remains valid if we replace the sequence $(y_l')_{l\in D}$ with the sequence $(y_l'')_{l \in D}$.

Claim 2. We have that

(a) $\quad |f(\sum\limits_{k\in D} \mu_k y_k')| \leq 73 \cdot \| \sum\limits_{k=1}^{2p_j} \mu_k t_k \|_{J_{T_0}}$.

(b) $\quad |f(\sum\limits_{k\in D} \mu_k y_k'')| \leq 73 \cdot \| \sum\limits_{k=1}^{2p_j} \mu_k t_k \|_{J_{T_0}}$.

Proof of Claim 2. We shall only show (a). The proof of (b) is almost identical; only minor modifications are required.

For each $a \in \mathcal{A}$ we set

$$D_a = \{k \in D : \ f_a \text{ covers } y_k'\}.$$

Setting $\mathcal{A}' = \{a \in \mathcal{A} : \ D_a \neq \emptyset\}$, we observe that $\mathcal{A}'$ is a complete subtree of the tree $\mathcal{A}$. We shall construct two families of functionals $(g_a)_{a\in\mathcal{A}'}$ and $(h_a)_{a \in \mathcal{A}'}$ such that the following conditions are satisfied for every $a \in \mathcal{A}'$.

(i) $\operatorname{supp} g_a \subset D_a$ and $\operatorname{supp} h_a \subset D_a$, while $\operatorname{supp} g_a \cap \operatorname{supp} h_a = \emptyset$.
(ii) $g_a \in D_0$ and $\|h_a\|_\infty \leq \frac{1}{m_{2i-1}}$.
(iii) $g_a(\sum\limits_{k\in D_a} \mu_k t_k) \geq 0$ and $h_a(\sum\limits_{k \in D_a} \mu_k t_k) \geq 0$.
(iv) $|f(\sum\limits_{k\in D_a} \mu_k y_k')| \leq (72 g_a + h_a)(\sum\limits_{k \in D_a} \mu_k t_k)$.

For $a \in \mathcal{A}'$ which is non maximal in $\mathcal{A}'$, we set $S'_a = S_a \cap \mathcal{A}' = \{\beta \in S_a : \ D_\beta \neq \emptyset\}$. Observe for later use, that the sets $(D_\beta)_{\beta \in S'_a}$ are successive and pairwise disjoint.

The construction of $(g_a)_{a\in\mathcal{A}'}$ and $(h_a)_{a\in\mathcal{A}'}$ is inductive. Let $a \in \mathcal{A}'$ and suppose that for every $\beta \in \mathcal{A}'$, $\beta \succ a$ the functionals g_β, h_β have been defined satisfying conditions (i), (ii), (iii), (iv). We distinguish the following cases.

Case 1. a is a maximal node of the tree $\mathcal{A}$.

Then f_a is of the form $f_a = e^*_{l_a}$, while the set D_a is a singleton, $D_a = \{k_a\}$. We set $g_a = \operatorname{sgn}(\mu_{k_a}) \cdot t^*_{k_a}$ and $h_a = 0$. Conditions (i), (ii), (iii) are obvious, while from Remark 7.5 and Lemma 7.2 we get that

$$|f_a(\sum_{k\in D_a} \mu_k y'_k)| = |\mu_{k_a}|\cdot|f_a(y'_{k_a})| \leq |\mu_{k_a}|\cdot\|y'_{k_a}\| \leq 36\cdot|\mu_{k_a}| \leq (72g_a+h_a)(\sum_{k\in D_a} \mu_k t_k).$$

Case 2. $w(f_a) \geq m_{2i+1}$.

Then from Remark 7.5 and Lemma 7.2 we get that $|f_a(y'_k)| \leq \frac{19}{m_{2i}}$ for every $k \in D_a$, thus, taking into account that $\#(D_a) \leq 2p_j$ and that from our choice of i, $38p_j < m_{2i-1}$, it follows that

$$|f_a(\sum_{k\in D_a} \mu_k y'_k)| \leq \max_{k\in D_a} |\mu_k| \cdot \frac{19 \cdot \#(D_a)}{m_{2i}} \leq \max_{k\in D_a} |\mu_k| \cdot \frac{1}{m_{2i-1}}.$$

We select $k_a \in D_a$ with $|\mu_{k_a}| = \max_{k\in D_a} |\mu_k|$ and we set $g_a = 0$ and $h_a = \operatorname{sgn}(\mu_{k_a}) \cdot \frac{1}{m_{2i-1}} \cdot e^*_{k_a}$.

Case 3. f_a is the result of an $(\mathcal{A}_{n_p}, \frac{1}{m_p})$ operation for some $p \leq 2i$.

Let $f_a = \frac{1}{m_p} \sum_{\beta\in S_a} f_\beta$ with $\#S_a \leq n_p$. We set $T_a = D_a \setminus \bigcup_{\beta\in S'_a} D_\beta$.

From Remark 7.7, for each $k \in T_a$ there exists $\beta_k \in S_a$ such that $\operatorname{supp}(f_{\beta_k}) \cap \operatorname{supp}(y'_k) \neq \emptyset$ and $\operatorname{supp}(f_{\beta_k}) \cap \operatorname{supp}(y'_l) = \emptyset$ for every $l \in D$, $l \neq k$. This implies that $\beta_k \in S_a \setminus S'_a$. Since clearly the correspondence

$$\begin{array}{ccc} T_a & \longrightarrow & S_a \setminus S'_a \\ k & \longmapsto & \beta_k \end{array}$$

is one to one, it follows that $\#T_a + \#S'_a \leq \#S_a \leq n_p$. We set

$$g_a = \frac{1}{m_p}\big(\sum_{\beta\in S'_a} g_\beta + \sum_{k\in T_a} \operatorname{sgn}(\mu_k)t^*_k\big) \quad \text{and} \quad h_a = \frac{1}{m_p}\sum_{\beta\in S'_a} h_\beta.$$

From our last observation and the inductive assumptions it follows that $g_a \in D_0$, while, again from our inductive assumptions, we have that $\|h_a\|_\infty \leq \frac{1}{m_{2i-1}}$ and $g_a(\sum_{k\in D_a} \mu_k t_k) \geq 0$, $h_a(\sum_{k\in D_a} \mu_k t_k) \geq 0$.

For every $k\in T_a$, Remark 7.5 and Lemma 7.2 yield that $|f_a(y'_k)|\leq\frac{54}{m_p}\leq\frac{72}{m_p}$. Therefore

$$\begin{aligned}|f_a(\sum_{k\in D_a}\mu_k y'_k)| &\leq \sum_{\beta\in S_a}\frac{1}{m_p}|f_\beta(\sum_{k\in D_\beta}\mu_k y'_k)|+\sum_{k\in T_a}|f_a(\mu_k y'_k)|\\ &\leq \sum_{\beta\in S'_a}\frac{1}{m_p}(72g_\beta+h_\beta)(\sum_{k\in D_\beta}\mu_k t_k)+\sum_{k\in T_a}|\mu_k|\cdot\frac{72}{m_p}\\ &= (72g_a+h_a)(\sum_{k\in D_a}\mu_k t_k).\end{aligned}$$

Case 4. f_a is a $2i+1$ special functional.
Let $f_a=\varepsilon_a\frac{1}{m_{2i}}E(\phi_1+\cdots\phi_{l_0}+\psi_{l_0+1}+\cdots+\psi_d)$, where $\phi_{l_0+1}\neq\psi_{l_0+1}$, $d\leq n_{2i+1}$ and $\max E=\max\operatorname{supp}\psi_d$. From the definition of the sets $F=\bigcup_{\beta\in\mathcal{B}}I_\beta$ and $D=\{k:\ k\notin F\text{ and }\operatorname{supp}(f)\cap\operatorname{supp}(y_k)\neq\emptyset\}$ we get that the set

$$R=\{k\in D_a:\ f_a\text{ covers }y'_k\text{ and }\operatorname{supp}E(\phi_1+\cdots+\phi_{l_0})\cap\operatorname{supp}(y'_k)\neq\emptyset\}$$

contains at most two elements (i.e. $\#R\leq 2$). We set

$$g_a=\frac{1}{2}\sum_{k\in R}\operatorname{sgn}(\mu_k)t^*_k.$$

We observe that $|f_a(\sum_{k\in R}\mu_k y'_k)|\leq 36\sum_{k\in R}|\mu_k|=72g_a(\sum_{k\in R}\mu_k t_k)$.

Since $y_k=\frac{m_{2i}}{M}\sum_{r=(k-1)M+1}^{kM}x_r$, the vector y'_k takes the form $y'_k=\frac{m_{2i}}{M}(x_{(k-1)M+1}+\cdots+x_{s-1}+x'_s)$ for some $s\leq kM$ where x'_s is of the form $x'_s=[\min\operatorname{supp}x_s,m]x_s$.

Let $k\in D_a\setminus R$. In order to give an upper estimate of the action of f_a on y'_k, we may assume, without loss of generality, that $x'_s=x_s$. Since $(x_r,\phi_r)_{r=1}^{n_{2i+1}}$ is a $(6,2i+1)$ dependent sequence we have that $w(\psi_l)\neq w(\phi_r)$ for all pairs (l,r) with $(l,r)\neq(l_0+1,l_0+1)$, while $|\psi_{l_0+1}(x_{l_0+1})|\leq\|x_{l_0+1}\|\leq 6$. It follows that

$$\begin{aligned}|f_a(y'_k)| &\leq \frac{m_{2i}}{M}\Big|(\sum_{l=l_0+1}^{d}\psi_l)(\sum_{r=(k-1)M+1}^{s}x_r)\Big|\\ &\leq \frac{m_{2i}}{M}\sum_{r=(k-1)M+1}^{s}\sum_{l=l_0+1}^{d}|\psi_l(x_r)|\\ &\leq \frac{m_{2i}}{M}\Big(6+\sum_{r=(k-1)M+1}^{s}(\sum_{w(\psi_l)<w(\phi_r)}\frac{18}{w(\psi_l)}+\sum_{w(\psi_l)>w(\phi_r)}\frac{6}{w(\phi_r)^2})\Big)\\ &\leq \frac{1}{m_{2i+1}}.\end{aligned}$$

Thus

$$|f_a(\sum_{k\in D_a\setminus R}\mu_k y'_k)|\leq\max_{k\in D_a\setminus R}|\mu_k|\cdot 2p_j\cdot\frac{1}{m_{2i+1}}\leq\max_{k\in D_a\setminus R}|\mu_k|\cdot\frac{1}{m_{2i}}.$$

We select $k_a \in D_a \setminus R$ such that $|\mu_{k_a}| = \max\limits_{k \in D_a \setminus R} |\mu_k|$ and we set $h_a = \mathrm{sgn}(\mu_{k_a}) \cdot \frac{1}{m_{2i}} \cdot t^*_{k_a}$.

We easily get that

$$|f_a(\sum_{k \in D_a} \mu_k y'_k)| \leq (72 g_a + h_a)(\sum_{k \in D_a} \mu_k t_k)$$

while inductive assumptions (i), (ii), (iii) are also satisfied for the functionals g_a, h_a.

Case 5. f_a is a $2q+1$ special functional for some $q < i$.

Let $f_a = \varepsilon_a \frac{1}{m_{2q}}(f_{\beta_1} + f_{\beta_2} + \cdots + f_{\beta_d})$, where $d \leq n_{2q+1} (\leq n_{2i-1})$. We set

$$l_0 = \min\{l : \max \operatorname{supp} f_{\beta_l} \geq \min \operatorname{supp} y'_1\}.$$

Then using our assumption that $\min \operatorname{supp} x_1 > m_{2i+1}$ (see the choice of the dependent sequence $(x_r, \phi_r)_{r=1}^{n_{2i+1}}$ in the beginning of the present section), the fact that $\min \operatorname{supp} y'_1 = \min \operatorname{supp} x_1$ and the definition of the special sequences, we get that

$$m_{2i+1} < \min \operatorname{supp} y'_1 \leq \max \operatorname{supp} f_{\beta_{l_0}} < w(f_{\beta_{l_0+1}}).$$

Thus for every k, using Lemma 7.2(ii) and Remark 7.5, we get that

$$\begin{aligned} \sum_{l=l_0+1}^{d} |f_{\beta_l}(y'_k)| &\leq \sum_{l=l_0+1}^{d} \left(\frac{18 m_{2i}}{w(f_{\beta_l})} + \frac{36 m_{2i}}{M}\right) \\ &\leq 18 m_{2i} \cdot \frac{2}{m_{2i+2}} + n_{2i-1} \cdot \frac{36 m_{2i}}{M} \\ &\leq \frac{2}{m_{2i}}. \end{aligned}$$

This yields that $\sum\limits_{k \in D_a} \sum\limits_{l=l_0+1}^{d} |f_{\beta_l}(y'_k)| \leq 2 p_j \cdot \frac{2}{m_{2i}} \leq \frac{1}{m_{2i-1}}$.

We observe that there exists at most one $k_0 \in D_a \setminus D_{\beta_{l_0}}$ such that $\operatorname{supp} f_{\beta_{l_0}} \cap \operatorname{supp} y'_{k_0} \neq \emptyset$. Without loss of generality, we assume that such a k_0 exists. We set

$$g_a = \frac{1}{2}(g_{\beta_{l_0}} + e^*_{k_0}).$$

We select $k_a \in D_a \setminus (D_{\beta_{l_0}} \cup \{k_0\})$ such that $|\mu_{k_a}| = \max\{|\mu_k| : k \in D_a \setminus (D_{\beta_{l_0}} \cup \{k_0\})\}$ and we set

$$h_a = \frac{1}{m_{2q}} h_{\beta_{l_0}} + \mathrm{sgn}(\mu_{k_a}) \cdot \frac{1}{m_{2i-1}} \cdot t^*_{k_a}.$$

Then conditions (i), (ii),(iii) are obviously satisfied, while

$$\begin{aligned}
|f_a(\sum_{k\in D_a}\mu_k y'_k)| &\leq \frac{1}{m_{2q}}\Big(\big|f_{\beta_{l_0}}(\sum_{k\in D_{\beta_{l_0}}}\mu_k y'_k)\big| + |f_{\beta_{l_0}}(\mu_{k_0}y'_{k_0})| \\
&\qquad + \sum_{k\in D_a}\sum_{l=l_0+1}^{d}|f_{\beta_l}(\mu_k y'_k)|\Big) \\
&\leq \frac{1}{m_{2q}}\Big(72\cdot g_{\beta_{l_0}}(\sum_{k\in D_{\beta_{l_0}}}\mu_k t_k) + h_{\beta_{l_0}}(\sum_{k\in D_{\beta_{l_0}}}\mu_k t_k) \\
&\qquad +36|\mu_{k_0}| + \frac{1}{m_{2i-1}}\Big) \\
&\leq (72g_a + h_a)(\sum_{k\in D_a}\mu_k t_k).
\end{aligned}$$

The inductive construction is complete.

For the functionals g_0, h_0 corresponding to the root $0 \in \mathcal{A}$ of the tree $\mathcal{A}$, and taking into account that $D_0 = D$, we get that

$$\begin{aligned}
|f(\sum_{k\in D}\mu_k y'_k)| &\leq 72\cdot g_0(\sum_{k\in D_0}\mu_k t_k) + h_0(\sum_{k\in D_0}\mu_k t_k) \\
&\leq 72\cdot g_0(\sum_{k=1}^{2p_j}\mu_k t_k) + \max_k|\mu_k|\cdot 2p_j\cdot\frac{1}{m_{2i-1}} \\
&\leq 72\cdot\|\sum_{k=1}^{2p_j}\mu_k t_k\|_{J_{T_0}} + \max_k|\mu_k| \\
&\leq 73\cdot\|\sum_{k=1}^{2p_j}\mu_k t_k\|_{J_{T_0}}.
\end{aligned}$$

The proof of the claim is complete. □

From Claim 1 and Claim 2 we conclude that

$$\begin{aligned}
|f(\sum_{k=1}^{2p_j}\mu_k y_k)| &\leq |f(\sum_{k\in F}\mu_k y_k)| + |f(\sum_{k\in D}\mu_k y'_k)| + |f(\sum_{k\in D}\mu_k y''_k)| \\
&\leq 3\cdot\|\sum_{k=1}^{2p_j}\mu_k t_k\|_{J_{T_0}} + 73\cdot\|\sum_{k=1}^{2p_j}\mu_k t_k\|_{J_{T_0}} + 73\cdot\|\sum_{k=1}^{2p_j}\mu_k t_k\|_{J_{T_0}} \\
&\leq 150\cdot\|\sum_{k=1}^{2p_j}\mu_k t_k\|_{J_{T_0}}.
\end{aligned}$$

This completes the proof of the right side inequality of (8) and also the proof of Theorem 6.4. □

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