SATURATED EXTENSIONS, THE ATTRACTORS METHOD AND HEREDITARILY JAMES TREE SPACES

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Dedicated to the memory of R.C. James

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0. Introduction

The purpose of the present work is to provide examples of HI Banach spaces with no reflexive subspace and study their structure. As is well known W.T. Gowers [G1] has constructed a Banach space $\mathfrak{X}_{gt}$ with a boundedly complete basis $(e_n)_n$, not containing ℓ_1, and such that all of its infinite dimensional subspaces have non separable dual. We shall refer to this space as the Gowers Tree space. The predual $(\mathfrak{X}_{gt})_*$, namely the space generated by the biorthogonal of the basis, also has the property that it does not contain c_0 or a reflexive subspace. It remains unknown whether $\mathfrak{X}_{gt}$ is HI and moreover the structure of $\mathcal{L}(\mathfrak{X}_{gt})$ is unclear. Notice that Gowers dichotomy [G2] yields that $\mathfrak{X}_{gt}$ and $(\mathfrak{X}_{gt})_*$ contain HI subspaces. The structure of $\mathfrak{X}^*_{gt}$ also remains unclear. The main obstacle for understanding the structure of $\mathfrak{X}_{gt}$ or $\mathcal{L}(\mathfrak{X}_{gt})$ is the use of a probabilistic argument for establishing the existence of vectors with certain properties.

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Our approach in constructing HI spaces with no reflexive subspace, is different from Gowers' one. In particular we avoid the use of any probabilistic argument and thus we are able to control the structure of the spaces as well as the structure of the spaces of bounded linear operators acting on them. Moreover we are able to provide examples of spaces X exhibiting a vast difference between the structures of X and X^*.

The following are the highlight of our results:

- There exists a HI Banach space X with a shrinking basis and with no reflexive subspace. Moreover every $T : X \to X$ is of the form $\lambda I + W$ with W weakly compact (and hence strictly singular).

The absence of reflexive subspaces in X in conjunction with the property that every strictly singular operator is weakly compact is evidence supporting the existence of Banach spaces such that every non Fredholm operator is compact.

- The dual X^* of the previous X is HI and reflexively saturated and the dual of every subspace Y of X is also HI.

This shows a strong divergence between the structure of X and X^*. We recall that in [AT2] a reflexive HI space X is constructed whose dual X^* is unconditionally saturated. The analogue of this in the present setting is the following one:

- There exists a HI Banach space Y with a shrinking basis and with no reflexive subspace, such that the dual space Y^* is reflexive and unconditionally saturated.

The definition of the space Y requires an adaptation of the methods of [AT2] within the present framework of building spaces with no reflexive subspace.

- There exists a partition of the basis $(e_n)_n$ of the previous X into two sets $(e_n)_{n\in L_1}$, $(e_n)_{n\in L_2}$ such that setting $X_{L_1} = \overline{\text{span}}\{e_n : \ n \in L_1\}$, $X_{L_2} = \overline{\text{span}}\{e_n : \ n \in L_2\}$, both $X^*_{L_1}$, $X^*_{L_2}$ are HI with no reflexive subspace.

The pairs $X_{L_i}, X^*_{L_i}$ for $i = 1, 2$ share similar properties with the pair $(\mathfrak{X}_{gt})_*$ and $\mathfrak{X}_{gt}$.

- The space X^{**} is non separable and every w^*-closed subspace of X^{**}, is either isomorphic to ℓ_2 or is non-separable and contains ℓ_2. Therefore every quotient of X^* has a further quotient isomorphic to ℓ_2. Moreover X^{**}/X is isomorphic to $\ell_2(\Gamma)$.

It seems also possible that $\mathfrak{X}^*_{gt}$ satisfies a similar to the above property although this is not easily shown. Further X^* is the first example of a HI space with the following property: X^*/Y is HI whenever Y is w^*-closed (this is equivalent to say that for every subspace Z of X, Z^* is HI) and also every quotient of X^* has a further quotient which is isomorphic to ℓ_2.

- There exists a non separable HI Banach space Z not containing a reflexive subspace and such that every bounded linear operator $T : Z \to Z$ is of the form $T = \lambda I + W$ with W a weakly compact (hence strictly singular) operator with separable range.

This is an extreme construction resulting from a variation of the methods used in the construction of the space X involved in the previous results. The fact of the matter is that these methods are not stable. Thus some minor changes in the initial data could produce spaces with entirely different structure. Notice that the space Z is of the form Y^{**} with Y and Y^* sharing similar properties with the pair X, X^* appearing in the previous statements.

We shall proceed to a more detailed presentation of the results of the paper and also of the methods used for constructing the spaces, which are interesting on their own. We have divided the rest of the introduction in three subsections. The first concerns the structure of Banach spaces not containing ℓ_1, c_0 or reflexive subspace. The second is devoted to saturated extensions and in the third we explain the method of attractors which permits the construction of dual pairs X, X^* with strongly divergent structure.

0.1. **Hereditarily James Tree spaces.** Separable spaces like Gowers Tree space undoubtedly have peculiar structure. Roughly speaking, in every subspace one can find a structure similar to the James tree basis. Next we shall attempt to be more precise. Thus we shall define the Hereditarily James Tree spaces, making more transparent their structure. We begin by recalling some of the fundamental characteristics of James' paradigm.

In the sequel we shall denote by $(\mathcal{D}, \prec)$ the dyadic tree and by $[\mathcal{D}]$, the set of all branches (or the body) of $\mathcal{D}$. As usual we would consider that the nodes of $\mathcal{D}$ consist of finite sequences of 0's and 1's and $a \prec b$ iff a is an initial part of b. The lexicographic order of $\mathcal{D}$, denoted by $\prec_{lex}$ defines a well ordering which is consistent with the tree order (i.e. $a \prec b$ yields that $a \prec_{lex} b$).

The space JT.

The James Tree space JT ([J]) is defined as the completion of $(c_{00}(\mathcal{D}), \|\cdot\|_{JT})$ where for $x \in c_{00}(\mathcal{D})$, $\|x\|_{JT}$ is defined as follows:

$$\|x\|_{JT} = \sup\Big\{\Big(\sum_{i=1}^{n}\Big(\sum_{n\in s_i} x(n)\Big)^2\Big)^{1/2} : \ (s_i)_{i=1}^n \text{ pairwise disjoint segments }\Big\}.$$

The main properties of the space JT, is that does not contain ℓ_1 and has nonseparable dual.

Next, we list some properties of JT related to our consideration.

- The Hamel basis $(e_a)_{a\in\mathcal{D}}$ of $c_{00}(\mathcal{D})$ ordered with the lexicographic order defines a (conditional) boundedly complete basis of JT.
- For every branch b in $[\mathcal{D}]$, $b = (a_1 \prec a_2 \prec \cdots \prec a_n \cdots)$ the sequence $(e_{a_n})_n$ is non trivial weak Cauchy and moreover $b^* = w^* - \sum_{n=1}^{\infty} e^*_{a_n}$ defines a norm one functional in JT^*.
- The biorthogonal functionals of the basis $(e^*_a)_{a\in\mathcal{D}}$ generate the predual JT_* of JT and they satisfy the following property.
  For every segment s of $\mathcal{D}$ setting $s^* = \sum_{a\in s} e^*_a$ we have that $\|s^*\| = 1$.

It is worth pointing out an alternative definition of the norm of JT. Thus we consider the following subset of $c_{00}(\mathcal{D})$,

$$G_{JT} = \Big\{\sum_{i=1}^{n} \lambda_i s^*_i : (s_i)_{i=1}^n \text{ are disjoint finite segments and } \sum_{i=1}^{n} \lambda_i^2 \leq 1\Big\}$$

Here s_i^* are defined as before. It is an easy exercise to see that the norm induced by the set G_{JT} on $c_{00}(\mathcal{D})$ coincides with the norm of JT.

The James Tree properties.

Let X be a space with a Schauder basis $(e_n)_n$. A block subspace Y of X has the boundedly complete (shrinking) James tree property if there exists a seminormalized block (in the lexicographical order $\prec_{lex}$ of $\mathcal{D}$) sequence $(y_a)_{a\in\mathcal{D}}$ in Y and a $c>0$ such that the following holds.

(1) **(boundedly complete)** There exists a bounded family $(b^*)_{b\in[\mathcal{D}]}$ in X^*, such that for each $b \in [\mathcal{D}]$, $b = (a_1, a_2, \ldots, a_n, \ldots)$ the sequence $(y_{a_n})_n$ is non trivial weakly Cauchy with $\lim b^*(y_{a_n}) > c$ and $\lim b_1^*(y_{a_n}) = 0$ for all $b_1 \neq b$.

(2) **(shrinking)** For all finite segments s of $\mathcal{D}$, $\| \sum_{a\in s} y_a \| \le c$.

Let's observe that $(e_a)_{a\in\mathcal{D}}$ in JT satisfies the boundedly complete James Tree property while $(e_a^*)_{a\in\mathcal{D}}$ in JT_* satisfies the shrinking one. Also, if the initial space X has a boundedly complete basis only the boundedly complete James Tree property could occur. A similar result holds if X has a shrinking basis. Finally if Y has the boundedly complete James Tree property, then Y^* is non separable and if X has a shrinking basis and Y has the (shrinking) James Tree property, then Y^{**} is non separable.

For simplicity, in the sequel we shall consider that the initial space X has either a boundedly complete or a shrinking basis. Thus if a block subspace has the James Tree property, then it will be determined as either boundedly complete or shrinking according to the corresponding property of the initial basis.

Definition 0.1. Let X be a Banach space with a Schauder basis.

(a) A family $\mathcal{L}$ of block subspaces of X has the James Tree property, provided every Y in $\mathcal{L}$ has that property.

(b) The space X is said to be Hereditarily James Tree (HJT) if it does not contain c_0, ℓ_1 and every block subspace Y of X, has the James Tree property.

It follows from Gowers' construction that the Gowers Tree space $\mathfrak{X}_{gt}$, and its predual $(\mathfrak{X}_{gt})_*$ are HJT spaces.

One of the results of the present work is that HJT property is not preserved under duality. Namely, there exists a HJT space X with a shrinking basis, such that X^* is reflexively (even unconditionally) saturated. However, in the same example there exists a subspace Y of X with Y^* also an HJT space.

One of the basic ingredients in our approach to building HJT spaces is the following space:

Proposition 0.2. There exists a space $JT_{\mathcal{F}_2}$ with a boundedly complete basis $(e_n)_n$ such that the following hold:

(i) The space $JT_{\mathcal{F}_2}$ is ℓ_2 saturated.

(ii) The basis $(e_n)_n$ is normalized weakly null and for every $M \in [\mathbb{N}]$ the subspace $X_M = \overline{\text{span}}\{e_n : n \in M\}$ has the James tree property.

It is clear that none subsequence $(e_n)_{n\in M}$ is unconditional. Thus the basis of $JT_{\mathcal{F}_2}$ shares similar properties with the classical Maurey Rosenthal example [MR]. We shall return to this space in the sequel explaining more about its structure and its difference from Gowers' space.

Codings and tree structures. As is well known, every attempt to impose tight (or conditional) structure in some Banach space, requires the definition of the conditional elements which in turn results from the existence of special sequences defined with the use of a coding. What is less well known is that the codings induce a tree structure in the special sequences. As we shall explain shortly, the James tree structure in the subspaces of HJT spaces, like $\mathfrak{X}_{gt}$, $(\mathfrak{X}_{gt})_*$ or the spaces presented in this paper, are directly related to codings.

Let's start with a general definition of a coding, and the obtained special sequences. Consider a collection $(F_j)_j$ with each F_j a countable family of elements of $c_{00}(\mathbb{N})$. To make more transparent the meaning of our definitions, let's assume that each $F_j = \{\frac{1}{m_j^2}\sum_{k\in F} e_k^* : \ F\subset\mathbb{N},\ \#F\leq n_j\}$ where $(m_j),(n_j)$ are appropriate fast increasing sequences of natural numbers. Notice that the elements of the family $\mathcal{T}=\cup_j F_j$ and the combinations of them will play the role of functionals belonging to a norming set. This explains the use of e_k^* instead of e_k. For simplicity, we also assume that the families $(F_j)_j$ are pairwise disjoint. This happens in the aforementioned example although it is not always true. Under this additional assumption to each $\phi\in\cup_j F_j$ corresponds a unique index by the rule $\text{ind}(\phi)=j$ iff $\phi\in F_j$. Further for a finite block sequence $s=(\phi_1,\dots,\phi_d)$ with each $\phi_i\in\cup_j F_j$, we define $\text{ind}(s)=\{\text{ind}(\phi_1),\dots,\text{ind}(\phi_d)\}$.

The σ-coding: Let Ω_1,Ω_2 be a partition of $\mathbb{N}$ into two infinite disjoint subsets. We denote by $\mathcal{S}$ the family of all block sequences $s=(\phi_1<\phi_2<\cdots<\phi_d)$ such that $\phi_i\in\cup_j F_j$, $\text{ind}(\phi_1)\in\Omega_1$, $\{\text{ind}(\phi_2)<\cdots<\text{ind}(\phi_d)\}\subset\Omega_2$. Clearly $\mathcal{S}$ is countable, hence there exists an injection

$$\sigma:\mathcal{S}\to\Omega_2$$

satisfying $\sigma(s)>\text{ind}(s)$ for every $s\in\mathcal{S}$.

The σ-special sequences: A sequence $s=(\phi_1<\phi_2<\cdots<\phi_n)$ in $\mathcal{S}$ is said to be a σ-special sequence iff for every $1\leq i<n$ setting $s_i=(\phi_1<\cdots<\phi_i)$ we have that

$$\phi_{i+1}\in F_{\sigma(s_i)}.$$

The following tree-like interference holds for σ-special sequences.

Let s,t be two σ-special sequences with $s=(\phi_1,\dots,\phi_n)$, $t=(\psi_1,\dots,\psi_m)$. We set

$$i_{s,t}=\max\{i:\text{ind}(\phi_i)=\text{ind}(\psi_i)\}$$

if the later set is non empty. Otherwise we set $i_{s,t}=0$. Then the following are easily checked.

(a) For every $i<i_{s,t}$ we have that $\phi_i=\psi_i$.
(b) $\{\text{ind}(\phi_i):i>i_{s,t}\}\cap\{\text{ind}(\psi_j):j>i_{s,t}\}=\emptyset$.

These two properties immediately yield that the set $\mathcal{T}\cup_j F_j$ endowed with the partial order $\phi\prec_\sigma\psi$ iff there exists a σ-special sequence $(\phi_1,\dots,\phi_n)$ and $1\leq i<j\leq n$ with $\phi=\phi_i$ and $\psi=\phi_j$ is a tree.

Now for the given tree structure $(\mathcal{T}, \prec_\sigma)$ we will define norms similar to the classical James tree norm mentioned above.

The space $JT_{\mathcal{F}_2}$: For the first application the family $(F_j)_j$ is the one defined above.

For a σ-special sequence $s = (\phi_1, \ldots, \phi_n)$ and an interval E of $\mathbb{N}$ we set $s^* = \sum_{k=1}^n \phi_k$ and let Es^* be the restriction of s^* on E (or the pointwise product $s^*\chi_E$). A σ-special functional x^* is any element Es^* as before.

Also, for a σ-special functional $x^* = Es^*$, $s = (\phi_1, \ldots, \phi_n)$, we let $\mathrm{ind}(x^*) = \{\mathrm{ind}(\phi_k) : \mathrm{supp}\,\phi_k \cap E \neq \emptyset\}$. We consider the following set

$$\mathcal{F}_2 = \{\pm e_n^* :\ n \in \mathbb{N}\} \cup \{\sum_{i=1}^d a_i x_i^* :\ a_i \in \mathbb{Q},\ \sum_{i=1}^d a_i^2 \leq 1,\ (x_i^*)_{i=1}^d \text{ are } \sigma\text{-special functionals with } (\mathrm{ind}(x_i^*))_{i=1}^d \text{ pairwise disjoint},\ d \in \mathbb{N}\}$$

The space $JT_{\mathcal{F}_2}$ is the completion of $(c_{00}, \|.\|_{\mathcal{F}_2})$ where for $x \in c_{00}$,

$$\|x\|_{\mathcal{F}_2} = \sup\{\phi(x) : \phi \in \mathcal{F}_2\}.$$

Comparing the norming set $\mathcal{F}_2$ with the norming set G_{JT} of JT one observes that σ-special functionals in $\mathcal{F}_2$ play the role of the functionals s^* defined by the segments of the dyadic tree $\mathcal{D}$. As we have mentioned in Proposition 0.2, the space $JT_{\mathcal{F}_2}$, like JT, is ℓ_2 saturated, but for every $M \in [\mathbb{N}]$, the subspace $X_M \overline{\mathrm{span}}\{e_n :\ n \in M\}$ has non separable dual. The later is a consequence of the fact that the tree structure $(\mathcal{T}, \prec_\sigma)$ is richer than that of the dyadic tree basis in JT. Indeed, it is easy to check that for every $M \in [\mathbb{N}]$ we can construct a block sequence $(\phi_a)_{a\in\mathcal{D}}$ such that

(i) $\phi_a = \frac{1}{m_{j_a}^2} \sum_{k\in F_a} e_k^*$ where $\#F_a = n_{j_a}$ and $F_a \subset M$, while $F_a < F_\beta$ if $a \prec_{lex} \beta$.

(ii) For a branch $b = (a_1 \prec a_2 \prec \cdots \prec a_n \prec \cdots)$ of $\mathcal{D}$ and for every $n \in \mathbb{N}$ we have that $(\phi_{a_1}, \ldots, \phi_{a_n})$ is a σ-special sequence.

Defining now $x_a = \frac{m_{j_a}^2}{n_{j_a}} \sum_{k\in F_a} e_k$, the family $(x_a)_{a\in\mathcal{D}}$ provides the James tree structure of X_M.

The Gowers Tree space. The definition of $\mathfrak{X}_{gt}$ uses similar ingredients with the corresponding of $JT_{\mathcal{F}_2}$ although structurally the two spaces are entirely different. The norming set G_{gt} of Gowers space is saturated under the operations $(\mathcal{A}_{n_j}, \frac{1}{m_j})_j$. We recall that a subset G of c_{00} is closed (or saturated) for the operation $(\mathcal{A}_n, \frac{1}{m})$ if for every $\phi_1 < \phi_2 < \cdots < \phi_k$, $k \leq n$ with $\phi_i \in G$, $i = 1, \ldots, k$, the functional $\phi = \frac{1}{m}\sum_{i=1}^k \phi_i$ belongs to G.

The norming set G_{gt} is the minimal subset of c_{00} satisfying the following conditions:

(i) $\{\pm e_k^* : k \in \mathbb{N}\} \subset G_{gt}$, G_{gt} is symmetric and closed under the operation of restricting elements to the intervals.

(ii) G_{gt} is closed in the $(\mathcal{A}_{n_j}, \frac{1}{m_j})_j$ operations. We also set

$$K_j = \{\phi \in G_{gt} : \phi \text{ is the result of a } (\mathcal{A}_{n_j}, \frac{1}{m_j}) \text{ operation}\}$$

(iii) G_{gt} contains the set

$$\Big\{\sum_{i=1}^{d} a_i x_i^* : a_i \in \mathbb{Q},\ \sum_{i=1}^{d} a_i^2 \leq 1,\ (x_i^*)_{i=1}^d,\ \sigma\text{-special functionals}$$
$$\text{with } (\mathrm{ind}(x_i^*))_{i=1}^d \text{ pairwise disjoint},\ d \in \mathbb{N}\Big\}$$

(iv) G_{gt} is rationally convex.

We explain briefly condition (iii). For a coding σ, the σ-special sequences $(\phi_1, \ldots, \phi_n)$ are defined as in the case of $\mathcal{F}_2$. Here the set K_j plays the role of the corresponding F_j in $\mathcal{F}_2$. The σ-special functionals x^*, are defined as in the case of $\mathcal{F}_2$.

Let's observe that G_{gt} is almost identical with $\mathcal{F}_2$, although the spaces defined by them are entirely different. The essential difference between $\mathcal{F}_2$ and G_{gt} is that in the case of $\mathcal{F}_2$ each F_j, $j \in \mathbb{N}$ does not norm any subspace of $JT_{\mathcal{F}_2}$, while in $\mathfrak{X}_{gt}$ each K_j defines an equivalent norm on $\mathfrak{X}_{gt}$. The later means that in every subspace Y of $\mathfrak{X}_{gt}$, the families K_j, $j \in \mathbb{N}$ as well as $\{x^* : x^* \text{ is a } \sigma\text{-special functional}\}$ and $\{\sum\limits_{i=1}^{d} \lambda_i x_i^* : \sum\limits_{i=1}^{n} \lambda_i^2 \leq 1,\ (\mathrm{ind}(x_i^*))_{i=1}^d \text{ are pairwise disjoint}\}$ define equivalent norms making it difficult to distinguish the action of them on the elements of Y. Thus, while the spaces of the form $JT_{\mathcal{F}_2}$ can be studied in terms of the classical theory, the space $\mathfrak{X}_{gt}$ requires advanced tools, like Gowers probabilistic argument, which do not permit a complete understanding of its structure.

0.2. **Saturated extensions.** The method of HI extensions appeared in the Memoirs monograph [AT1] and was used to derive the following two results:

- Every separable Banach space Z not containing ℓ_1 is a quotient of a separable HI space X, with the additional property that Q^*Z^* is a complemented subspace of X^*. (Here Q denotes the quotient map from X to Z.)
- There exists a nonseparable HI Banach space.

Roughly speaking, the method of HI extensions provides a tool to connect a given norm, usually defined through a norming set G with a HI norm. The resulting new norm will preserve some of the ingredients of the initial norm and will also be HI. To some extent, HI extensions, have similar goals with HI interpolations ([AF]) and some of the results could be obtained with both methods. However it seems that the method of extensions is very efficient when we want to construct dual pairs X, X^* with divergent structure. This actually requires the combination of extensions with the method of attractors, which appeared in [AT2] where a reflexive HI space X is constructed with X^* unconditionally saturated.

In the sequel we shall provide a general definition of saturated extensions which include several forms of extensions which appeared elsewhere (c.f. [AT1, AT2, ArTo])

Let $\mathcal{M}$ be a compact family of finite subsets of $\mathbb{N}$. For the purposes of the present paper, $\mathcal{M}$ will be either some $\mathcal{A}_n = \{F \subset \mathbb{N} : \#F \leq n\}$, or some $\mathcal{S}_n$, the n^{th} Schreier family. For $0 < \theta < 1$, the $(\mathcal{M}, \theta)$-operation on c_{00} is a map which assigns to each $\mathcal{M}$-admissible block sequence $(\phi_1 < \phi_2 < \cdots < \phi_n)$, the functional $\theta \sum_{i=1}^{n} \phi_i$. (We recall that $\phi_1, \phi_2, \ldots, \phi_n$ is $\mathcal{M}$-admissible if $\{\min \operatorname{supp} \phi_i : \ i = 1, \ldots, n\}$ belongs to $\mathcal{M}$.) A subset G of c_{00} is said to be closed in the $(\mathcal{M}, \theta)$-operation, if for every $\mathcal{M}$-admissible block sequence $\phi_1, \ldots, \phi_n$, with each $\phi_i \in G$, the functional $\theta \sum_{i=1}^{n} \phi_i$ belongs to G. When we refer to saturated norms we shall mean that there exists a norming set G which is closed under certain $(\mathcal{M}_j, \theta_j)_j$ operations.

Let G be a subset of c_{00}. The set G is said to be a ground set if it is symmetric, $\{e_n^* : \ n \in \mathbb{N}\}$ is contained in G, $\|\phi\|_\infty \leq 1$, $\phi(n) \in \mathbb{Q}$ for all $\phi \in G$ and G is closed under the restriction of its elements to intervals of $\mathbb{N}$. A ground norm, $\|\cdot\|_G$ is any norm induced on c_{00} by a ground set G. It turns out that for every space $(X, \|\cdot\|_X)$ with a normalized Schauder basis $(x_n)_n$ there exists a ground set G_X such that the natural map $e_n \mapsto x_n$ defines an isomorphism between $(X, \|\cdot\|_X)$ and $\overline{(c_{00}, \|\cdot\|_{G_X})}$.

Saturated extensions of a ground set G. Let G be a ground set, $(m_j)_j$ an appropriate sequence of natural numbers and $(\mathcal{M}_j)_j$ a sequence of compact families such that $(\mathcal{M}_j)_j$ is either $(\mathcal{A}_{n_j})_j$ or $(S_{n_j})_j$.

Denote by E_G the minimal subset of c_{00} such that

(i) The ground set G is a subset of E_G.
(ii) The set E_G is closed in the $(\mathcal{M}_j, \frac{1}{m_j})$ operation.
(iii) The set E_G is rationally convex.

Definition 0.3. A subset D_G of E_G is said to be a saturated extension of the ground set G if the following hold:

(i) The set D_G is a subset of E_G, the ground set G is contained in D_G and D_G is closed under restrictions of its elements to intervals.
(ii) The set D_G is closed under even operations $(\mathcal{M}_{2j}, \frac{1}{m_{2j}})_j$.
(iii) The set D_G is rationally convex.
(iv) Every $\phi \in D_G$ admits a tree analysis $(f_t)_{t\in T}$ with each $f_t \in D_G$.

Denoting by $\|\cdot\|_{D_G}$ the norm on c_{00} induced by D_G and letting X_{D_G} be the space $\overline{(c_{00}, \|\cdot\|_{D_G})}$, we call X_{D_G} a *saturated extension* of the space $X_G = \overline{(c_{00}, \|\cdot\|_G)}$.

Let's point out that the basis $(e_n)_n$ of c_{00} is a bimonotone boundedly complete Schauder basis of X_{D_G} and that the identity $I : X_{D_G} \to X_G$ is a norm one operator. Observe also that we make no assumption concerning the odd operations. As we will see later making several assumptions for the odd operations, we will derive saturated extensions with different properties.

A last comment on the definition of D_G, is related to the condition (iv). The tree analysis $(f_t)_{t\in T}$ of a functional f in E_G describes an inductive procedure for obtaining f starting from elements of the ground set G and either applying operations $(\mathcal{M}_j, \frac{1}{m_j})$ or taking rational convex combinations. This tree structure is completely irrelevant to the tree structures discussed in the previous subsection. Its role is to help estimate upper bounds of the norm of vectors in X_{D_G}.

Properties and variants of Saturated extensions.

As we have mentioned, for $x \in c_{00}$, $\|x\|_G \leq \|x\|_{D_G}$. This is an immediate consequence of the fact that $G \subset D_G$. On the other hand, there are cases of ground sets G such that D_G does not add more information beyond G itself. Such a case is when G defines a norm $\|\cdot\|_G$ equivalent to the ℓ_1 norm. The measure of the fact that $\|\cdot\|_{D_G}$ is strictly greater than $\|\cdot\|_G$ on a subspace Y of X_{D_G} is that the identity operator $I : X_{D_G} \to X_G$ restricted to Y is a strictly singular one. If $I : X_{D_G} \to X_G$ is strictly singular, then we refer to strictly singular extensions. The first result we want to mention is that strictly singular extensions are reflexive ones. More precisely the following holds:

Proposition 0.4. Let Y be a closed subspace of X_{D_G} such that $I|_Y : Y \to X_G$ is strictly singular. Then Y is reflexively saturated. In particular X_{D_G} is reflexively saturated whenever it is a strictly singular extension.

Next we proceed to specify the odd operations and to derive additional information on the structure of X_{D_G} whenever X_{D_G} is a strictly singular extension.

(a) Unconditionally saturated extensions.

This is the case where $D_G = E_G = D_G^u$. In this case the following holds:

Proposition 0.5. Let Y be a closed subspace of $X_{D_G^u}$ such that $I|_Y : Y \to X_G$ is strictly singular. Then Y is unconditionally (and reflexively) saturated.

(b) Hereditarily Indecomposable extensions.

HI extensions, are the most important ones. In this case the norming set D_G^{hi} is defined as follows. D_G^{hi} is the minimal subset of c_{00} satisfying the following conditions

(i) $\{e_n^* : n \in \mathbb{N}\} \subset D_G^{hi}$, D_G^{hi} is symmetric and closed under restriction of its elements to intervals.

(ii) D_G^{hi} is closed under $(\mathcal{M}_{2j}, \frac{1}{m_{2j}})_j$ operations.

(iii) For each j, D_G^{hi} is closed under $(\mathcal{M}_{2j-1}, \frac{1}{m_{2j-1}})$ operation on $2j-1$ special sequences.

(iv) D_G^{hi} is rationally convex.

The $2j-1$ special sequences are defined through a coding σ and satisfy the following conditions.

(a) $(f_1, \dots, f_d)$ is $\mathcal{M}_{2j-1}$ admissible

(b) For $i \leq i \leq d$ there exists some $j \in \mathbb{N}$ such that $f_i \in K_{2j} = \{\frac{1}{m_{2j}} \sum_{l=1}^{k} \phi_l : \phi_1 < \cdots < \phi_k$ is $\mathcal{M}_{2j}$ admissible, $\phi_l \in D_G^{hi}\}$ and if $i > 1$ then $2j = \sigma(f_1, \dots, f_{i-1})$.

Notice that in the definition of D_G^{hi} we do not refer to the tree analysis. The reason is that the existence of a tree analysis follows from the minimality of D_G^{hi} and the conditions involved in its definition.

The analogue of the previous results also holds for HI extensions.

Proposition 0.6. Let Y be a closed subspace of $X_{D_G^{hi}}$ such that $I|_Y : Y \to X_G$ is strictly singular. Then Y is a HI space. In particular strictly singular and HI extensions yield HI spaces.

The above three propositions indicate that if we wish to have additional structure on X_{D_G}, $X_{D_G^u}$, $X_{D_G^{hi}}$ we need to consider strictly singular extensions. As is shown in [AT1], this is always possible. Indeed, for every ground set G such that the corresponding space X_G does not contain ℓ_1 there exists a family $(\mathcal{M}_j, \frac{1}{m_j})_j$ such that the saturated extension of G by this family is a strictly singular one. Thus the following is proven ([AT1]).

Theorem 0.7. Let X be a Banach space with a normalized Schauder basis $(x_n)_n$ such that X contains no isomorphic copy of ℓ_1. Then there exists a HI space Z with a normalized basis $(z_n)_n$ such that the map $z_n \mapsto x_n$ has a linear extension to a bounded operator $T : Z \to X$.

This theorem in conjunction with the following one yields that every separable Banach space X not containing ℓ_1 is the quotient of a HI space.

Theorem 0.8 ([AT1])**.** Let X be a separable Banach space not containing ℓ_1. Then there exists a space Y not containing ℓ_1, with a normalized Schauder basis $(y_n)_n$ and a bounded linear operator $T : Y \to X$ such that $(Ty_n)_n$ is a dense subset of the unit sphere of X.

The predual $(X_{D_G^{hi}})_*$**.** As we have mentioned before the basis $(e_n)_{n\in\mathbb{N}}$ of $X_{D_G^{hi}}$ is boundedly complete, hence the space $(X_{D_G^{hi}})_*$, which is the subspace of $X^*_{D_G^{hi}}$ norm generated by the biorthogonal functionals $(e_n^*)_{n\in\mathbb{N}}$, is a predual of $X_{D_G^{hi}}$. In many cases it is shown that $(X_{D_G^{hi}})_*$ is also a HI space. This requires some additional information concerning the weakly null block sequences in X_G, which is stronger than the assumption that the identity map $I : X_{D_G^{hi}} \to X_G$ is strictly singular. For example in [AT1], for extensions using the operations $(\mathcal{S}_{n_j}, \frac{1}{m_j})_j$, had been assumed that the ground set G is $\mathcal{S}_2$ bounded. In the present paper for the operations $(\mathcal{A}_{n_j}, \frac{1}{m_j})_j$ we introduce the concept of strongly strictly singular extension which yields that $(X_{D_G^{hi}})_*$ is HI. It is also worth pointing out that $(X_{D_G^{hi}})_*$ is not necessarily reflexively saturated as happens for the strictly singular extensions X_{D_G} $X_{D_G^{hi}}$. This actually will be a key point in our approach for constructing HI spaces with no reflexive subspace.

0.3. **The attractors method.** Let's return to our initial goal, namely constructing HI spaces with no reflexive subspace. It is clear from our preceding discussion that HI extensions of ground sets G such that X_G does not contain ℓ_1 yield reflexively saturated HI spaces. Therefore there is no hope to obtain HI spaces with no reflexive subspace as a result of a HI extension of a ground set G. As mentioned in [ArTo] saturation and HI methods share common metamathematical ideas with the forcing method in set theory. In particular the fact that HI extensions are reflexively saturated is similar to the well known collapse phenomena in the extensions of models of set theory via the forcing method. An illustrating example of such phenomena in HI extensions is that $\mathfrak{X}_{gt}$ is a quotient of a HI and reflexively saturated space. In spite of all these discouraging observations we claim that HI extensions could help to yield HI spaces with no reflexive subspace, and this is closely related to the structure of $(X_{D_G^{hi}})_*$. Evidently from the initial stages of HI theory, ([GM1],[GM2],[AD]) and for many years, a norming set D was defined, using saturation methods and codings, in such a way as to impose certain properties in the space $(X, \|\cdot\|_D)$. In [AT2] the norming set D was designed to impose divergent properties in $(X, \|\cdot\|_D)$ and $(X, \|\cdot\|_D)^*$. The method used for this is the attractors method, not so named in [AT2], which will also be used in the present work.

The general principle of the attractors method is the following:

We are interested in designing a ground set G and a HI extension D_G^{hi} such that the following two divergent properties hold:

(a) $X_{D_G^{hi}}$ is a strictly singular extension of X_G. In other words every subspace Y of $X_{D_G^{hi}}$ contains a further subspace Z on which the G-norm becomes negligible.
(b) The set G is asymptotic in $(X_{D_G^{hi}})_*$. This means that there exists $c > 0$ such that for every subspace Y of $(X_{D_G^{hi}})_*$ and every $\varepsilon > 0$ there exists $\phi \in G$ with $\|\phi\|_{(X_{D_G^{hi}})_*} \geq c$ and $\mathrm{dist}(\phi, Y) < \varepsilon$.

In other words, we want G to be small, as a norming set for the space $X_{D_G^{hi}}$ and large as a subset of $(X_{D_G^{hi}})_*$. Notice that such a relation between G and D_G^{hi} requires the two sets to be built with similar materials, and moreover to impose certain special functionals in D_G^{hi} (we call these attractor functionals) which will allow us to attract in every subspace of $(X_{D_G^{hi}})_*$ part of the structure of the set G.

Let us be more precise explaining how we define the corresponding sets G and D_G^{hi} to obtain a HI extension $X_{D_G^{hi}}$, such that $(X_{D_G^{hi}})_*$ is also HI and does not contain reflexive subspaces.

The ground set $\mathcal{F}_2$ and the norming set $D_{\mathcal{F}_2}$. We start by defining the following family $(F_j)_j$. We shall use the sequence of positive integers $(m_j)_j$, $(n_j)_j$ recursively defined as follows:

- $m_1 = 2$ and $m_{j+1} = m_j^5$.
- $n_1 = 4$, and $n_{j+1} = (5n_j)^{s_j}$ where $s_j = \log_2 m_{j+1}^3$.

We set $F_0 = \{\pm e_n^* : \ n \in \mathbb{N}\}$ and for $j = 1, 2, \dots$ we set

$$F_j = \Big\{\frac{1}{m_{4j-3}^2} \sum_{i\in I} \pm e_i^* : \ \#(I) \leq \frac{n_{4j-3}}{2}\Big\} \cup \big\{0\big\}.$$

Using the family $(F_j)_j$ and a coding $\sigma_{\mathcal{F}}$, we define the ground set $\mathcal{F}_2$ in the same manner as in the first subsection.

Next we define the set $D_{\mathcal{F}_2}$ which is a HI extension of $\mathcal{F}_2$ with attractors as follows:

The set $D_{\mathcal{F}_2}$ is a minimal subset of c_{00} satisfying the following properties:

(i) $\mathcal{F}_2 \subset D_{\mathcal{F}_2}$, $D_{\mathcal{F}_2}$ is symmetric (i.e. if $f \in D_{\mathcal{F}_2}$ then $-f \in D_{\mathcal{F}_2}$) and $D_{\mathcal{F}_2}$ is closed under the restriction of its elements to intervals of $\mathbb{N}$ (i.e. if $f \in D_{\mathcal{F}_2}$ and E is an interval of $\mathbb{N}$ then $Ef \in D_{\mathcal{F}_2}$).

(ii) $D_{\mathcal{F}_2}$ is closed under $(\mathcal{A}_{n_{2j}}, \frac{1}{m_{2j}})$ operations, i.e. if $f_1 < f_2 < \cdots < f_{n_{2j}}$ belong to $D_{\mathcal{F}_2}$ then the functional $f = \frac{1}{m_{2j}}(f_1 + f_2 + \cdots + f_{n_{2j}})$ belongs also to $D_{\mathcal{F}_2}$.

(iii) $D_{\mathcal{F}_2}$ is closed under $(\mathcal{A}_{n_{4j-1}}, \frac{1}{m_{4j-1}})$ operations on special sequences i.e. for every n_{4j-1} special sequence $(f_1, f_2, \dots, f_{n_{4j-1}})$ the functional $f = \frac{1}{m_{4j-1}}(f_1 + f_2 + \cdots + f_{n_{4j-1}})$ belongs to $D_{\mathcal{F}_2}$. In this case we say that f is a **special functional**.

(iv) $D_{\mathcal{F}_2}$ is closed under $(\mathcal{A}_{n_{4j-3}}, \frac{1}{m_{4j-3}})$ operations on attractor sequences i.e. for every $4j-3$ attractor sequence $(f_1, f_2, \dots, f_{n_{4j-3}})$ the functional $f = \frac{1}{m_{4j-3}}(f_1 + f_2 + \cdots + f_{n_{4j-3}})$ belongs to $D_{\mathcal{F}_2}$. In this case we say that f is an **attractor**.

(v) The set $D_{\mathcal{F}_2}$ is rationally convex.

In the above definition, the special functionals and the attractors, defined in (iii) and (iv) respectively, require some more explanation. First, the n_{4j-1} special sequences $(f_1, \dots, f_{n_{4j-1}})$ are defined through a coding σ as in the case of the aforementioned HI extensions. Thus each f_i, $1 \leq i \leq n_{4j-1}$ belongs to some

$$K_{2j} = \{\frac{1}{m_{2j}} \sum_{l=1}^{n_{2j}} \phi_l : \ \phi_1 < \cdots < \phi_{n_{2j}}, \ \phi_l \in D_{\mathcal{F}_2}\}$$

where $2j$ is equal to $\sigma(f_1, \dots, f_{i-1})$ whenever $1 < i$.

The special functionals will determine the HI property of the extension $D_{\mathcal{F}_2}$.

Each $4j-3$ attractor sequence $f_1 < \cdots < f_{n_{4j-3}}$ is of the following form. All the odd members of the sequence are elements of $\cup_j K_{2j}$ while the even members are $f_{2i} = e^*_{\ell_{2i}}$ and furthermore the sequence $f_1, \dots, f_{n_{4j-3}}$ is determined by the coding σ in a similar manner to the n_{4j-1} special sequence. Let us observe that for every $j \in \mathbb{N}$ there exist many $P \subset \mathbb{N}$ with the following properties. First $\#P \frac{n_{4j-3}}{2}$, hence $\frac{1}{m^2_{4j-3}} \sum_{\ell \in P} e^*_\ell \in F_j$ and also there exists an attractor sequence $(f_1, \dots, f_{4j-3})$ with $\{e^*_\ell : \ell \in P\}$ coinciding with the even terms of the sequence. The purpose of the attractors is to make the family $\cup_j F_j$ asymptotic in the space $(X_{D_{\mathcal{F}_2}})_*$. In particular using attractors, the following is proved.

For every subspace Y of $(X_{D_{\mathcal{F}_2}})_*$ and every $j \in \mathbb{N}$ there exist $\phi_j \in Y$ and $\psi_j = \frac{1}{m^2_{4j-3}} \sum_{\ell \in P} e^*_\ell \in F_j$, such that

(a) $\|\phi_j + \psi_j\| > \frac{1}{144}$.

(b) $\|\phi_j - \psi_j\| \leq \frac{1}{m_{4j-3}}$.

This shows that indeed $\cup_j F_j$ is asymptotic and furthermore we can copy a complete dyadic block subtree of $(\mathcal{T} = \cup_j F_j, \prec_{\sigma_{\mathcal{F}}})$ into the subspace Y which yields that Y is not reflexive.

The following summarizes the main steps in our approach to constructing HI spaces with no reflexive subspace.

(1) For two appropriately chosen sequences $(m_j)_j$, $(n_j)_j$ we set $F_j = \{\frac{1}{m_{4j-3}^2}\sum_{k\in F} e_k^* : \#F \leq \frac{n_{4j-3}}{2}\}$ and for the family $(F_j)_j$ we construct the norming set $\mathcal{F}_2$ and the James Tree space $JT_{\mathcal{F}_2}$.

(2) The space $JT_{\mathcal{F}_2}$ does not contain ℓ_1 and for every weakly null sequence $(x_n)_n$ in $JT_{\mathcal{F}_2}$ with $\|x_n\| \leq C$, $\lim \|x_n\|_\infty = 0$ and every $m \in \mathbb{N}$ there exists $L \in [\mathbb{N}]$ such that for every $y^* \in \mathcal{F}_2$

$$\#\{n \in L : |y^*(x_n)| \geq \frac{1}{m}\} \leq 66m^2C^2. \tag{1}$$

(3) We consider the HI extension with attractors $D_{\mathcal{F}_2}^{hi}$ of $\mathcal{F}_2$ defined by the operations $(\mathcal{A}_{n_j}, \frac{1}{m_j})$, and we denote by $\mathfrak{X}_{\mathcal{F}_2}$ the space $\overline{(c_{00}, \|\cdot\|_{D_{\mathcal{F}_2}^{hi}})}$.

(4) Inequality (1) yields that $\mathfrak{X}_{\mathcal{F}_2}$ is a strongly strictly singular extension of $JT_{\mathcal{F}_2}$. Therefore:
   (i) The space $\mathfrak{X}_{\mathcal{F}_2}$ is HI and reflexively saturated.
   (ii) The predual $(\mathfrak{X}_{\mathcal{F}_2})_*$ is HI.

(5) Using the attractor functionals, we copy into every subspace of $(\mathfrak{X}_{\mathcal{F}_2})_*$ a complete dyadic subtree of $(\mathcal{T}, \prec_{\mathcal{F}})$ which shows that $(\mathfrak{X}_{\mathcal{F}_2})_*$ is a Hereditarily James Tree space (HJT) and hence it does not contain a reflexive subspace.

Notice that $(\mathfrak{X}_{\mathcal{F}_2})_*$ shares with the space X, in the statements presented at the beginning of the introduction, most of the properties stated there. However for some of the properties a variation is required. In fact there exists a complete subtree $(\mathcal{T}', \prec_{\sigma_F})$ of $(\mathcal{T}, \prec_{\sigma_F})$ such that for the corresponding space $\mathfrak{X}_{\mathcal{F}_2'}$ we have that $(\mathfrak{X}_{\mathcal{F}_2'})^*/(\mathfrak{X}_{\mathcal{F}_2'})_* = \ell_2(\Gamma)$ with $\#\Gamma = 2^\omega$. The space $(\mathfrak{X}_{\mathcal{F}_2'})_*$ coincides with X in the aforementioned statements.

The construction of a nonseparable HI space Z not containing reflexive subspaces requires changing the framework with the operations $(\mathcal{S}_{n_j}, \frac{1}{m_j})_j$ instead of $(\mathcal{A}_{n_j}, \frac{1}{m_j})_j$. In this framework the set $\mathcal{F}_s$ and the space $JT_{\mathcal{F}_s}$ are defined. More precisely $\mathcal{F}_s$ is defined in a similar manner as $\mathcal{F}_2$ based on the families

$$F_j = \{\frac{1}{m_{4j-3}^2}\sum_{i\in I} \pm e_i^* : \ I \in \mathcal{S}_{n_{4j-3}}\}.$$

Using the coding $\sigma_{\mathcal{F}}$, we define the special functionals and their indices as in the $\mathcal{F}_2$ case. Finally we set

$$\begin{aligned}\mathcal{F}_s \quad = \quad & \{\pm e_n^* : \ n \in \mathbb{N}\} \cup \{\sum_{i=1}^{d} x_i^* : \ \min\operatorname{supp} x_i^* \geq d, \ i = 1, \dots, n, \\ & (\operatorname{ind}(x_i^*))_{i=1}^d \text{ are pairwise disjoint}\}\end{aligned}$$

The HI extension with attractors is defined similarly to the $\mathfrak{X}_{\mathcal{F}_2}$ case. Then $\mathfrak{X}_{\mathcal{F}_s}$ is an asymptotic ℓ_1 and reflexively saturated HI space and also $(\mathfrak{X}_{\mathcal{F}_s})_*$ is HI while not containing any reflexive subspace. Passing to a complete subtree $(T', \prec_{\sigma_{\mathcal{F}}})$ of $(T, \prec_{\sigma_{\mathcal{F}}})$ and to the corresponding $\mathcal{F}_s'$, $\mathfrak{X}_{\mathcal{F}_s'}$, we obtain the additional property that $\mathfrak{X}_{\mathcal{F}_s'}^*/(\mathfrak{X}_{\mathcal{F}_s'})_* \cong c_0(\Gamma)$ with $\#\Gamma = 2^\omega$. As is shown in [AT1], this yields that $\mathfrak{X}_{\mathcal{F}_s'}^*$ is HI and since it contains a subspace $((\mathfrak{X}_{\mathcal{F}_s'})_*)$ with no reflexive subspace, the space $\mathfrak{X}_{\mathcal{F}_s'}^*$ has the same property.

Let's mention also that an HI asymptotic ℓ_1 Banach space X not containing a reflexive subspace, with nonseparable dual X^* which is also HI not containing any reflexive subspace, has been constructed in [AGT]. This space is the analogue of $\mathfrak{X}_{gt}$ in the frame of the operations $(\mathcal{S}_{n_j}, \frac{1}{m_j})_j$.

The last variant we present, concerns the HI space Y with a shrinking basis, not containing a reflexive subspace, such that the dual Y^* is unconditionally and reflexively saturated.

For this, starting with the set $\mathcal{F}_2$ we pass to an extension only with attractors and additionally we subtract a large portion of the conditional structure of the attractors. This permits us to show that the extension space $\mathfrak{X}^{us}_{\mathcal{F}_2}$ is unconditionally saturated. The remaining part of the conditional structure of the attractors, forces the predual $(\mathfrak{X}^{us}_{\mathcal{F}_2})_*$ to be HI and not to contain any reflexive subspace.

The paper is organized as follows. The first two sections are devoted to the presentation of the strictly singular and strongly strictly singular (HI) extensions with attractors of a ground space Y_G. We shall denote these as X_G. For the results presented in these two sections the attractors play no role. Thus all statements remain valid whether we consider extensions with attractors or not. The strictly singular extension, as they have defined before, provide information about X_G and $\mathcal{L}(X_G)$. In particular the following is proven:

Theorem 0.9. If X_G is a strictly singular extension (with or without attractors) then the natural basis of X_G is boundedly complete, the space X_G is HI, reflexively saturated and every T in $\mathcal{L}(X_G)$ is of the form $T = \lambda I + S$ with S a strictly singular operator.

Strongly strictly singular extensions concern the HI property of $(X_G)_*$ and the structures of $\mathcal{L}(X_G)$, $\mathcal{L}((X_G)_*)$. The following theorem includes the main results of Section 2.

Theorem 0.10. If X_G is a strongly strictly singular extension (with or without attractors) then in addition to the above we have the following

(i) The predual $(X_G)_*$ is HI.
(ii) Every strictly singular S in $\mathcal{L}(X_G)$ is weakly compact.
(iii) Every T in $\mathcal{L}((X_G)_*)$ is of the form $T = \lambda I + S$ with S being a strictly singular and weakly compact operator.

The following result concerning the quotients of X_G is also proved in Section 2.

Theorem 0.11. If X_G is a strongly strictly singular extension (with or without attractors) and Z is a w^* closed subspace of X_G then the quotient X_G/Z is HI.

The dual form of the above theorem is the following. For every subspace W of $(X_G)_*$ the dual space W^* is HI. Notice also that the additional assumption that Z is w^* closed can not be dropped, as the results of Section 5 indicate.

In Section 3 we study the spaces $JT_{\mathcal{F}_2}$. We are mainly concerned with proving the aforementioned (1) yielding that the extension $\mathfrak{X}_{\mathcal{F}_2}$ is a strongly strictly singular one. In Section 4 using attractors we prove that $(\mathfrak{X}_{\mathcal{F}_2})_*$ is a Hereditarily James Tree (HJT) space and hence it does not contain any reflexive subspace. Section 5 is devoted to the study of $(\mathfrak{X}_{\mathcal{F}_2})^*$. It is shown that for every subspace Y of $(\mathfrak{X}_{\mathcal{F}_2})_*$, the space ℓ_2 is isomorphic to a subspace of the nonseparable space Y^{**}. We also describe the definition of $\mathfrak{X}_{\mathcal{F}_2'}$ which has the additional property that $(\mathfrak{X}_{\mathcal{F}_2'})^*/(\mathfrak{X}_{\mathcal{F}_2'})_*$ is isomorphic to $\ell_2(\Gamma)$. Section 6 and Section 7 contain the variants $\mathfrak{X}_{\mathcal{F}_s}$, $\mathfrak{X}_{\mathcal{F}_2}^{us}$ mentioned before. We have also included two appendices. In Appendix A we present a proof of a form of the basic inequality used for estimating upper bounds for the action of functionals on certain vectors. In Appendix B we proceed to a systematic study of the James Tree spaces $JT_{\mathcal{F}_2}$, $JT_{\mathcal{F}_s}$ and $JT_{\mathcal{F}_{2,s}}$. We actually show that $JT_{\mathcal{F}_2}$ is ℓ_2 saturated while $JT_{\mathcal{F}_s}$ and $JT_{\mathcal{F}_{2,s}}$ are c_0 saturated. The study of James Tree spaces in Section 3 and Appendix B is not related to HI techniques and uses classical Banach space theory with Ramsey' s theorem also playing a key role.

1. Strictly singular extensions with attractors

In this section we introduce the ground sets G and then we define the extensions $X_G = T[G, (\mathcal{A}_{n_j}, \frac{1}{m_j})_j, \sigma]$ with low complexity saturation methods. Attractors are also defined. We provide conditions yielding the HI property of the extension X_G and we study the space of the operators $\mathcal{L}(X_G)$. The results and the techniques are analogue to the corresponding of [AT1] where extensions using higher complexity saturation methods are presented. We refer the reader to [ArTo] for an exposition of low complexity extensions. We also point out that the attractors in the present and next section will be completely neutralized. Their role will be revealed in Section 4 where we study the structure of $(\mathfrak{X}_{\mathcal{F}_2})_*$.

Definition 1.1. (ground sets) A set $G \subset c_{00}(\mathbb{N})$ is said to be ground if the following conditions are satisfied

(i) $e_n^* \in G$ for $n = 1, 2, \ldots$, G is symmetric (i.e. if $g \in G$ then $-g \in G$), and closed under restriction of its elements to intervals of $\mathbb{N}$ (i.e. if $g \in G$ and E is an interval of $\mathbb{N}$ then $Eg \in G$).

(ii) $\|g\|_\infty \leq 1$ for every $g \in G$ and $g(n) \in \mathbb{Q}$ for every $g \in G$ and $n \in \mathbb{N}$.

(iii) Denoting by $\| \ \|_G$ the ground norm on $c_{00}(\mathbb{N})$ defined by the rule $\|x\|_G = \sup\{g(x) : \ g \in G\}$, the ground space Y_G, which is the completion of $(c_{00}(\mathbb{N}) , \| \ \|_G)$ contains no isomorphic copy of ℓ_1.

It follows readily that the standard basis $(e_n)_n$ of Y_G is a bimonotone Schauder basis. The converse is also true. Namely if Y has a bimonotone basis $(y_n)_n$ and ℓ_1 does not embed into Y then there exists a ground set G such that Y_G is isometric to Y. Also, as is well known, every space $(Y, \| \ \|)$ with a basis $(e_n)_{n\in\mathbb{N}}$ admits an equivalent norm $|||\cdot|||$ such that $(e_n)_{n\in\mathbb{N}}$ is a bimonotome basis for $(Y, ||| \ |||)$.

Definition 1.2. (HI extensions with attractors) We fix two strictly increasing sequences of even positive integers $(m_j)_{j\in\mathbb{N}}$ and $(n_j)_{j\in\mathbb{N}}$ defined as follows:

- $m_1 = 2$ and $m_{j+1} = m_j^5$.

- $n_1 = 4$, and $n_{j+1} = (5n_j)^{s_j}$ where $s_j = \log_2 m_{j+1}^3$.

We let D_G be the minimal subset of $c_{00}(\mathbb{N})$ satisfying the following conditions:

(i) $G \subset D_G$, D_G is symmetric (i.e. if $f \in D_G$ then $-f \in D_G$) and D_G is closed under the restriction of its elements to intervals of $\mathbb{N}$ (i.e. if $f \in D_G$ and E is an interval of $\mathbb{N}$ then $Ef \in D_G$).

(ii) D_G is closed under $(\mathcal{A}_{n_{2j}}, \frac{1}{m_{2j}})$ operations, i.e. if $f_1 < f_2 < \cdots < f_{n_{2j}}$ belong to D_G then the functional $f = \frac{1}{m_{2j}}(f_1 + f_2 + \cdots + f_{n_{2j}})$ also belongs to D_G. In this case we say that the functional f is the result of an $(\mathcal{A}_{n_{2j}}, \frac{1}{m_{2j}})$ operation.

(iii) D_G is closed under $(\mathcal{A}_{n_{4j-1}}, \frac{1}{m_{4j-1}})$ operations on special sequences, i.e. for every n_{4j-1} special sequence $(f_1, f_2, \dots, f_{n_{4j-1}})$, the functional $f = \frac{1}{m_{4j-1}}(f_1 + f_2 + \cdots + f_{n_{4j-1}})$ belongs to D_G. In this case we say that f is a result of an $(\mathcal{A}_{n_{4j-1}}, \frac{1}{m_{4j-1}})$ operation and that f is a **special functional**.

(iv) D_G is closed under $(\mathcal{A}_{n_{4j-3}}, \frac{1}{m_{4j-3}})$ operations on attractor sequences, i.e. for every $4j-3$ attractor sequence $(f_1, f_2, \dots, f_{n_{4j-3}})$, the functional $f = \frac{1}{m_{4j-3}}(f_1 + f_2 + \cdots + f_{n_{4j-3}})$ belongs to D_G. In this case we say that f is a result of an $(\mathcal{A}_{n_{4j-3}}, \frac{1}{m_{4j-3}})$ operation and that f is an **attractor**.

(v) The set D_G is rationally convex.

The space $X_G = T[G, (\mathcal{A}_{n_j}, \frac{1}{m_j})_j, \sigma]$, which is the completion of the space $(c_{00}(\mathbb{N})\ , \|\ \|_{D_G})$, is called a **strictly singular extension with attractors** of the space Y_G, provided that the identity operator $I : X_G \to Y_G$ is strictly singular.

The norm satisfies the following implicit formula.

$$\begin{aligned}\|x\| \quad &= \quad \max\Big\{\|x\|_G,\ \sup_j\{\sup\frac{1}{m_{2j}}\sum_{i=1}^{n_{2j}}\|E_i x\|\},\\ &\qquad \sup\{\phi(x):\ \phi \text{ special functional}\},\ \sup\{\phi(x):\ \phi \text{ attractor}\}\Big\}\end{aligned}$$

where the inside supremum in the second term is taken over all choices $(E_i)_{i=1}^{n_{2j}}$ of successive intervals of $\mathbb{N}$.

We next complete the definition of the norming set D_G by giving the precise definition of special functionals and attractors.

From the minimality of D_G it follows that each $f \in D_G$ has one of the following forms.

(i) $f \in G$. We then say that f is of type 0.

(ii) $f = \pm Eh$ where h is a result of an $(\mathcal{A}_{n_j}, \frac{1}{m_j})$ operation and E is an interval. In this case we say that f is of type I. Moreover we say that the integer m_j is a weight of f and we write $w(f) = m_j$. We notice that an $f \in D_G$ may have many weights.

(iii) f is a rational convex combination of type 0 and type I functionals. In this case we say that f is of type II.

Definition 1.3. **(σ coding, special sequences and attractor sequences)** Let $\mathbb{Q}_s$ denote the set of all finite sequences $(\phi_1, \phi_2, \dots, \phi_d)$ such that $\phi_i \in c_{00}(\mathbb{N})$, $\phi_i \neq 0$ with $\phi_i(n) \in \mathbb{Q}$ for all i, n and $\phi_1 < \phi_2 < \cdots < \phi_d$. We fix a pair Ω_1, Ω_2 of disjoint infinite subsets of $\mathbb{N}$. From the fact that $\mathbb{Q}_s$ is countable we are able to define a Gowers-Maurey type injective coding function $\sigma : \mathbb{Q}_s \to \{2j : \ j \in \Omega_2\}$ such that $m_{\sigma(\phi_1,\phi_2,\dots,\phi_d)} > \max\{\frac{1}{|\phi_i(e_l)|} : l \in \operatorname{supp}\phi_i, \ i = 1, \dots, d\} \cdot \max\operatorname{supp}\phi_d$. Also, let $(\Lambda_i)_{i\in\mathbb{N}}$ be a sequence of pairwise disjoint infinite subsets of $\mathbb{N}$ with $\min\Lambda_i > m_i$.

(A) A finite sequence $(f_i)_{i=1}^{n_{4j-1}}$ is said to be a n_{4j-1} **special sequence** provided that
   (i) $(f_1, f_2, \dots, f_{n_{4j-1}}) \in \mathbb{Q}_s$ and $f_i \in D_G$ for $i = 1, 2, \dots, n_{4j-1}$.
   (ii) $w(f_1) = m_{2k}$ with $k \in \Omega_1$, $m_{2k}^{1/2} > n_{4j-1}$ and for each $1 \leq i < n_{4j-1}$, $w(f_{i+1}) = m_{\sigma(f_1,\dots,f_i)}$.

(B) A finite sequence $(f_i)_{i=1}^{n_{4j-3}}$ is said to be a n_{4j-3} **attractor sequence** provided that
   (i) $(f_1, f_2, \dots, f_{n_{4j-3}}) \in \mathbb{Q}_s$ and $f_i \in D_G$ for $i = 1, 2, \dots, n_{4j-3}$.
   (ii) $w(f_1) = m_{2k}$ with $k \in \Omega_1$, $m_{2k}^{1/2} > n_{4j-3}$ and $w(f_{2i+1}) = m_{\sigma(f_1,\dots,f_{2i})}$ for each $1 \leq i < n_{4j-3}/2$.
   (iii) $f_{2i} = e^*_{l_{2i}}$ for some $l_{2i} \in \Lambda_{\sigma(f_1,\dots,f_{2i-1})}$, for $i = 1, \dots, n_{4j-3}/2$.

The definition of the special functionals and the attractors completes the definition of the norming set D_G of the space X_G.

Remarks 1.4. (i) Since the sequence $(\frac{n_{2j}}{m_{2j}})_j$ increases to infinity and the norming set D_G is closed in the $(\mathcal{A}_{n_{2j}}, \frac{1}{m_{2j}})$ operations we get that the Schauder basis $(e_n)_{n\in\mathbb{N}}$ of X_G is boundedly complete.

(ii) The special sequences, as in previous constructions (see for example [GM1],[G2],[AD],[AT1]), are responsible for the absence of unconditionality in every subspace of X_G.

(iii) The attractors do not effect the results of the present section. Their role is to attract the structure of G, for certain G, in every subspace of the predual $(X_G)_* = \overline{\operatorname{span}}\{e^*_n : \ n \in \mathbb{N}\}$ of the space X_G. So, if we discard condition (iv) in the definition of the norming set D_G (Definition 1.2) then the corresponding space X_G, which we call a **strictly singular extension** provided that the identity operator $I : X_G \to Y_G$ is strictly singular, shares all the properties we shall prove in this section with those X_G's which are strictly singular extensions with attractors.

Definition 1.5. **(Rapidly increasing sequences)** A block sequence (x_k) in X_G is said to be a (C, ε) rapidly increasing sequence (R.I.S.), if $\|x_k\| \leq C$, and there exists a strictly increasing sequence (j_k) of positive integers such that

(a) $(\max\operatorname{supp} x_k)\frac{1}{m_{j_{k+1}}} < \varepsilon$.

(b) For every $k = 1, 2, \dots$ and every $f \in D_G$ with $w(f) = m_i$, $i < j_k$ we have that $|f(x_k)| \leq \frac{C}{m_i}$.

Remark 1.6. A subsequence of a (C,ε) R.I.S. remains a (C,ε) R.I.S. while a sequence which is a (C,ε) R.I.S. is also a (C',ε') R.I.S. if $C' \geq C$ and $\varepsilon' \geq \varepsilon$.

Proposition 1.7. Let $(x_k)_{k=1}^{n_{j_0}}$ be a (C,ε) R.I.S. with $\varepsilon \leq \frac{2}{m_{j_0}^2}$ such that for every $g \in G$, $\#\{k : \ |g(x_k)| > \varepsilon\} \leq n_{j_0 - 1}$. Then

1) For every $f \in D_G$ with $w(f) = m_i$,

$$|f(\frac{1}{n_{j_0}}\sum_{k=1}^{n_{j_0}} x_k)| \leq \begin{cases} \frac{3C}{m_{j_0} m_i}\,, & \text{if } \ i < j_0 \\ \frac{C}{n_{j_0}} + \frac{C}{m_i} + C\varepsilon\,, & \text{if } \ i \geq j_0 \end{cases}$$

In particular $\|\frac{1}{n_{j_0}}\sum\limits_{k=1}^{n_{j_0}} x_k\| \leq \frac{2C}{m_{j_0}}$.

2) If $(b_k)_{k=1}^{n_{j_0}}$ are scalars with $|b_k| \leq 1$ such that

$$|h(\sum_{k\in E} b_k x_k)| \leq C(\max_{k\in E}|b_k| + \varepsilon\sum_{k\in E}|b_k|) \tag{2}$$

for every interval E of positive integers and every $h \in D_G$ with $w(h) = m_{j_0}$, then

$$\|\frac{1}{n_{j_0}}\sum_{k=1}^{n_{j_0}} b_k x_k\| \leq \frac{4C}{m_{j_0}^2}.$$

The proof of the above proposition is based on what we call the basic inequality (see also [AT1], [ALT]). Its proof is presented in Appendix A.

Remark 1.8. The validity of Proposition 1.7 is independent of the assumption that the operator $I : X_G \to Y_G$ is strictly singular.

In the present section we shall prove several properties of the space X_G provided that the space X_G is a strictly singular extension of Y_G.

Definition 1.9. (exact pairs) A pair (x,ϕ) with $x \in X_G$ and $\phi \in D_G$ is said to be a (C,j,θ) exact pair (where $C \geq 1$, $j \in \mathbb{N}$, $0 \leq \theta \leq 1$) if the following conditions are satisfied:

(i) $1 \leq \|x\| \leq C$, for every $\psi \in D_G$ with $w(\psi) = m_i$, $i \neq j$ we have that $|\psi(x)| \leq \frac{2C}{m_i}$ if $i < j$, while $|\psi(x)| \leq \frac{C}{m_j^2}$ if $i > j$ and $\|x\|_\infty \leq \frac{C}{m_j^2}$.

(ii) ϕ is of type I with $w(\phi) = m_j$.

(iii) $\phi(x) = \theta$ and $\operatorname{ran} x = \operatorname{ran}\phi$.

Definition 1.10. (ℓ_1^k averages) Let $k \in \mathbb{N}$. A finitely supported vector $x \in X_G$ is said to be a $C - \ell_1^k$ average if $\|x\| > 1$ and there exist $x_1 < \ldots < x_k$ with $\|x_i\| \leq C$ such that $x = \frac{1}{k}\sum\limits_{i=1}^{k} x_i$.

Lemma 1.11. Let $j \in \mathbb{N}$ and $\varepsilon > 0$. Then every block subspace of X_G contains a vector x which is a $2 - \ell_1^{n_{2j}}$ average. If X_G is a strictly singular extension (with or without attractors) then we may select x to additionally satisfy $\|x\|_G < \varepsilon$.

Proof. If the identity operator $I : X_G \to Y_G$ is strictly singular we may pass to a further block subspace on which the restriction of I has norm less than

$\frac{\varepsilon}{2}$. For the remainder of the proof in this case, and the proof in the general case, we refer to [GM1] (Lemma 3) or to [AM]. □

Lemma 1.12. Let x be a $C-\ell_1^k$ average. Then for every $n\leq k$ and every sequence of intervals $E_1<\ldots<E_n$, we have that $\sum\limits_{i=1}^{n}\|E_ix\|\leq C(1+\frac{2n}{k})$. In particular if x is a $C-\ell_1^{n_j}$ average then for every $f\in D_G$ with $w(f)=m_i$, $i<j$ then $|f(x)|\leq\frac{1}{m_i}C(1+\frac{2n_{j-1}}{n_j})\leq\frac{3C}{2}\frac{1}{m_i}$.

We refer to [S] or to [GM1] (Lemma 4) for a proof.

Remark 1.13. Let $(x_k)_k$ be a block sequence in X_G such that each x_k is a $\frac{2C}{3}-\ell_1^{n_{j_k}}$ average and let $\varepsilon>0$ be such that $\#(\mathrm{ran}(x_k))\frac{1}{m_{j_{k+1}}}<\varepsilon$. Then Lemma 1.12 yields that condition (b) in the definition of R.I.S. (Definition 1.5) is also satisfied hence $(x_k)_k$ is a (C,ε) R.I.S. In this case we shall call $(x_k)_k$ a (C,ε) R.I.S. of ℓ_1 averages. From this observation and Lemma 1.11 it follows that if X_G is a strictly singular extension of Y_G then for every $\varepsilon>0$, every block subspace of X_G contains a $(3,\varepsilon)$ R.I.S. of ℓ_1 averages $(x_k)_{k\in\mathbb{N}}$ with $\|x_k\|_G<\varepsilon$.

Proposition 1.14. Suppose that X_G is a strictly singular extension of Y_G (with or without attractors). Let Z be a block subspace of X_G, let $j\in\mathbb{N}$ and let $\varepsilon>0$. Then there exists a $(6,2j,1)$ exact pair (x,ϕ) with $x\in Z$ and $\|x\|_G<\varepsilon$.

Proof. From the fact that the identity operator $I:X_G\to Y_G$ is strictly singular we may assume, passing to a block subspace of Z, that $\|z\|_G<\frac{\varepsilon}{6}\|z\|$ for every $z\in Z$. We choose a $(3,\frac{1}{n_{2j}})$ R.I.S. of ℓ_1 averages in Z, $(x_k)_{k=1}^{n_{2j}}$. For $k=1,2,\ldots,n_{2j}$ we choose $\phi_k\in D_G$ with $\mathrm{ran}\,\phi_k=\mathrm{ran}\,x_k$ such that $\phi_k(x_k)>1$. We set $\phi=\frac{1}{m_{2j}}\sum\limits_{k=1}^{n_{2j}}\phi_k$. We have that $\eta=\phi(\frac{m_{2j}}{n_{2j}}\sum\limits_{k=1}^{n_{2j}}x_k)>1$. On the other hand Proposition 1.7 yields that $\|\frac{m_{2j}}{n_{2j}}\sum\limits_{k=1}^{n_{2j}}x_k\|\leq 6$. We set $x=\frac{1}{\eta}\frac{m_{2j}}{n_{2j}}\sum\limits_{k=1}^{n_{2j}}x_k$.

We have that $1=\phi(x)\leq\|x\|\leq 6$, hence also $\|x\|_G\leq\varepsilon$, while $\mathrm{ran}\,\phi=\mathrm{ran}\,x$. From Proposition 1.7 it follows that for every $\psi\in D_G$ with $w(\psi)=m_i$, $i\neq 2j$ we have that $|\psi(x)|\leq\frac{9}{m_i}$ if $i<2j$ while $|\psi(x)|\leq m_{2j}(\frac{3}{n_{2j}}+\frac{3}{m_i}+\frac{3}{n_{2j}})\leq\frac{1}{m_{2j}^2}$ if $i>m_{2j}$. Finally $\|x\|_\infty\leq\frac{m_{2j}}{n_{2j}}\max\limits_k\|x_k\|_\infty\leq\frac{3m_{2j}}{n_{2j}}<\frac{1}{m_{2j}^2}$.

Therefore (x,ϕ) is a $(6,2j,1)$ exact pair with $x\in Z$ and $\|x\|_G<\varepsilon$. □

Definition 1.15. (dependent sequences and attracting sequences)

(A) A double sequence $(x_k,x_k^*)_{k=1}^{n_{4j-1}}$ is said to be a $(C,4j-1,\theta)$ dependent sequence (for $C>1$, $j\in\mathbb{N}$, and $0\leq\theta\leq 1$) if there exists a sequence $(2j_k)_{k=1}^{n_{4j-1}}$ of even integers such that the following conditions are fulfilled:

(i) $(x_k^*)_{k=1}^{n_{4j-1}}$ is a $4j-1$ special sequence with $w(x_k^*)=m_{2j_k}$ for each k.

(ii) Each (x_k, x_k^*) is a $(C, 2j_k, \theta)$ exact pair.

(B) A double sequence $(x_k, x_k^*)_{k=1}^{n_{4j-3}}$ is said to be a $(C, 4j-3, \theta)$ attracting sequence (for $C > 1$, $j \in \mathbb{N}$, and $0 \le \theta \le 1$) if there exists a sequence $(2j_k)_{k=1}^{n_{4j-3}}$ of even integers such that the following conditions are fulfilled:

(i) $(x_k^*)_{k=1}^{n_{4j-3}}$ is a $4j-3$ attractor sequence with $w(x_{2k-1}^*) = m_{2j_{2k-1}}$ and $x_{2k}^* = e_{l_{2k}}^*$ where $l_{2k} \in \Lambda_{2j_{2k}}$ for all $k \le n_{4j-3}/2$.

(ii) $x_{2k} = e_{l_{2k}}$.

(iii) Each (x_{2k-1}, x_{2k-1}^*) is a $(C, 2j_{2k-1}, \theta)$ exact pair.

Remark 1.16. If $(x_k, x_k^*)_{k=1}^{n_{4j-1}}$ is a $(C, 4j-1, \theta)$ dependent sequence (resp. $(x_k, x_k^*)_{k=1}^{n_{4j-3}}$ is a $(C, 4j-1, \theta)$ attracting sequence) then the sequence $(x_k)_k$ is a $(2C, \frac{1}{n_{4j-1}^2})$ R.I.S. (resp. a $(2C, \frac{1}{n_{4j-3}^2})$ R.I.S.). Let examine this for a $(C, 4j-3, \theta)$ attracting sequence (the proof for a dependent sequence is similar). First $\|x_k\| = \|e_{l_k}\| = 1$ if k is even while $\|x_k\| \le C$ if k is odd, as follows from the fact that (x_k, x_k^*) is a $(C, 2j_k, \theta)$ exact pair.

Second, the growth condition of the coding function σ in Definition 1.3 and condition (ii) in the same definition yield that for each k we have that $(\max\operatorname{supp} x_k)\frac{1}{m_{2j_{k+1}}} = \max\operatorname{supp} x_k^* \cdot \frac{1}{m_{\sigma(x_1^*,\dots,x_k^*)}}$
$< \min\{|x_i^*(e_l)| : \ l \in \operatorname{supp} x_i^*, \ i = 1, \dots, k\} \le \frac{1}{m_{2j_1}} < \frac{1}{n_{4j-3}^2}$.

Finally, if $f \in D_G$ with $w(f) = m_i$, $i < 2j_k$ then $|f(x_k)| = |f(e_{l_k})| \le \|f\|_\infty \le \frac{1}{m_i}$ if k is even, while $|f(x_k)| \le \frac{2C}{m_i}$ if k is odd, since in this case (x_k, x_k^*) is a $(C, 2j_k, \theta)$ exact pair.

Proposition 1.17. (i) Let $(x_k, x_k^*)_{k=1}^{n_{4j-1}}$ be a $(C, 4j-1, \theta)$ dependent sequence such that $\|x_k\|_G \le \frac{2}{m_{4j-1}^2}$ for $1 \le k \le n_{4j-1}$. Then we have that

$$\|\frac{1}{n_{4j-1}} \sum_{k=1}^{n_{4j-1}} (-1)^{k+1} x_k\| \le \frac{8C}{m_{4j-1}^2}.$$

(ii) If $(x_k, x_k^*)_{k=1}^{n_{4j-3}}$ is a $(C, 4j-3, \theta)$ attracting sequence with $\|x_{2k-1}\|_G \le \frac{2}{m_{4j-3}^2}$ for $1 \le k \le n_{4j-3}/2$ and for every $g \in G$ we have that $\#\{k : |g(x_{2k})| > \frac{2}{m_{4j-3}^2}\} \le n_{4j-4}$ then

$$\|\frac{1}{n_{4j-3}} \sum_{k=1}^{n_{4j-3}} (-1)^{k+1} x_k\| \le \frac{8C}{m_{4j-3}^2}.$$

(iii) If $(x_k, x_k^*)_{k=1}^{n_{4j-1}}$ is a $(C, 4j-1, 0)$ dependent sequence with $\|x_k\|_G \le \frac{2}{m_{4j-1}^2}$ for $1 \le k \le n_{4j-1}$ then we have that

$$\|\frac{1}{n_{4j-1}} \sum_{k=1}^{n_{4j-1}} x_k\| \le \frac{8C}{m_{4j-1}^2}.$$

Proof. The conclusion will follow from Proposition 1.7 2) after showing that the required conditions are fulfilled. We shall only show (i); the proof of (ii) and (iii) is similar.

From the previous remark the sequence $(x_k)_{k=1}^{n_{4j-1}}$ is a $(2C, \frac{1}{n_{4j-1}^2})$ R.I.S. hence it is a $(2C, \frac{2}{m_{4j-1}^2})$ R.I.S. (see Remark 1.6). It remains to show that for $f \in D_G$ with $w(f) = m_{4j-1}$ and for every interval E of positive integers we have that

$$|f\big(\sum_{k\in E}(-1)^{k+1}x_k\big)| \leq 2C(1 + \frac{2}{m_{4j-1}^2}\#(E)).$$

Such an f is of the form $f = \frac{1}{m_{4j-1}}(Fx_{t-1}^* + x_t^* + \cdots + x_r^* + f_{r+1} + \cdots + f_d)$ for some $4j-1$ special sequence $(x_1^*, x_2^*, \ldots, x_r^*, f_{r+1}, \ldots, f_{n_{4j-1}})$ where $x_{r+1}^* \neq f_{r+1}$ with $w(x_{r+1}^*) = w(f_{r+1})$, $d \leq n_{4j-1}$ and F is an interval of the form $[m, \max\operatorname{supp} x_{t-1}^*]$.

We estimate the value $f(x_k)$ for each k.

- If $k < t-1$ we have that $f(x_k) = 0$.
- If $k = t-1$ we get $|f(x_{t-1})| = \frac{1}{m_{4j-1}}|Fx_{t-1}^*(x_{t-1})| \leq \frac{1}{m_{4j-1}}\|x_{t-1}\| \leq \frac{C}{m_{4j-1}}$.
- If $k \in \{t, \ldots, r\}$ we have that $f(x_k) = \frac{1}{m_{4j-1}}x_k^*(x_k) = \frac{\theta}{m_{4j-1}}$.
- If $k > r+1$, then the injectivity of the coding function σ and the definition of special functionals yield that $w(f_i) \neq m_{2j_k}$ for all $i \geq r+1$. Using the fact that (x_k, x_k^*) is a $(C, 2j_k, \theta)$ exact pair and taking into account that $n_{4j-1}^2 < m_{2j_1} \leq \sqrt{m_{2j_k}}$ we get that

$$\begin{aligned} |f(x_k)| &= \frac{1}{m_{4j-1}}|(f_{r+1} + \ldots + f_d)(x_k)| \\ &\leq \frac{1}{m_{4j-1}}\Big(\sum_{w(f_i)<m_{2j_k}}|f_i(x_k)| + \sum_{w(f_i)>m_{2j_k}}|f_i(x_k)|\Big) \\ &\leq \frac{1}{m_{4j-1}}\Big(\sum_{4j-1<l<2j_k}\frac{2C}{m_l} + n_{4j-1}\frac{C}{m_{2j_k}^2}\Big) \\ &\leq \frac{C}{m_{4j-1}^2} \end{aligned}$$

- For $k = r+1$, using a similar argument to the previous case we get that $|f(x_{r+1})| \leq \frac{C}{m_{4j-1}} + \frac{C}{m_{4j-1}^2} < \frac{2C}{m_{4j-1}}$.

Let E be an interval. From the previous estimates we get that

$$\begin{aligned} |f\big(\sum_{k\in E}(-1)^{k+1}x_k\big)| &\leq |f(x_{t-1})| + \Big|\sum_{k\in E\cap[t,r]}\frac{\theta}{m_{4j-1}}(-1)^{k+1}\Big| \\ &\quad + |f(x_{r+1})| + \sum_{k\in E\cap(r+1,n_{4j-1}]}|f(x_k)| \\ &\leq \frac{C}{m_{4j-1}} + \frac{1}{m_{4j-1}} + \frac{C+1}{m_{4j-1}} + \frac{C}{m_{4j-1}^2}\#(E) \\ &< 2C(1 + \frac{2}{m_{4j-1}^2}\#(E)). \end{aligned}$$

The proof of the proposition is complete. □

Theorem 1.18. If the space X_G is a strictly singular extension, (with or without attractors) then it is Hereditarily Indecomposable.

Proof. Let Y and Z be a pair of block subspaces of X_G and let $\delta > 0$. We choose $j \in \mathbb{N}$ with $m_{4j-1} > \frac{48}{\delta}$. Using Proposition 1.14 we inductively construct a $(6, 4j-1, 1)$ dependent sequence $(x_k, x_k^*)_{k=1}^{n_{4j-1}}$ with $x_{2k-1} \in Y$, $x_{2k} \in Z$ and $\|x_k\|_G < \frac{1}{m_{4j-1}^2}$ for all k. From Proposition 1.17 (i) we get that $\|\frac{1}{n_{4j-1}} \sum_{k=1}^{n_{4j-1}} (-1)^{k+1} x_k\| \leq \frac{48}{m_{4j-1}^2}$. On the other hand the functional $x^* = \frac{1}{m_{4j-1}} \sum_{k=1}^{n_{4j-1}} x_k^*$ belongs to D_G and the estimate of x^* on the vector $\frac{1}{n_{4j-1}} \sum_{k=1}^{n_{4j-1}} x_k$ gives that $\|\frac{1}{n_{4j-1}} \sum_{k=1}^{n_{4j-1}} x_k\| \geq \frac{1}{m_{4j-1}}$.

Setting $y = \sum_{k=1}^{n_{4j-1}/2} x_{2k-1}$ and $z = \sum_{k=1}^{n_{4j-1}/4} x_{2k-1}$ we get that $y \in Y$, $z \in Z$ and $\|y - z\| < \delta \|y + z\|$. Therefore the space X_G is Hereditarily Indecomposable. $\square$

Proposition 1.19. If X_G is a strictly singular extension (with or without attractors) then the dual X_G^* of the space $X_G = T[G, (\mathcal{A}_{n_j}, \frac{1}{m_j})_j, \sigma]$ is the norm closed linear span of the w^* closure of G.

$$X_G^* = \overline{\text{span}}(\overline{G}^{w^*}).$$

Proof. Assume the contrary. Then setting $Z = \overline{\text{span}}(\overline{G}^{w^*})$ there exist $x^* \in X_G^* \setminus Z$ with $\|x^*\| = 1$ and $x^{**} \in X_G^{**}$ such that $Z \subset \ker x^{**}$, $\|x^{**}\| = 2$ and $x^{**}(x^*) = 2$. The space X_G contains no isomorphic copy of ℓ_1, since X_G is a HI space, thus from the Odell-Rosenthal theorem there exist a sequence $(x_k)_{k\in\mathbb{N}}$ in X_G with $\|x_k\| \leq 2$ such that $x_k \xrightarrow{w^*} x^{**}$. Since each e_n^* belongs to Z we get that $\lim_k e_n^*(x_k) = 0$ for all n, thus, using a sliding hump argument, we may assume that $(x_k)_{k\in\mathbb{N}}$ is a block sequence. Since also $x^*(x_k) \to x^{**}(x^*) = 2$ we may also assume that $1 < x^*(x_k)$ for all k. Let's observe that every convex combination of $(x_k)_{k\in\mathbb{N}}$ has norm greater than 1.

Considering each x_k as a continuous function $x_k : \overline{G}^{w^*} \to \mathbb{R}$ we have that the sequence $(x_k)_{k\in\mathbb{N}}$ is uniformly bounded and tends pointwise to 0, hence it is a weakly null sequence in $C(\overline{G}^{w^*})$. Since Y_G is isometric to a subspace of $C(\overline{G}^{w^*})$ we get that $x_k \xrightarrow{w} 0$ in Y_G thus there exists a convex block sequence $(y_k)_{k\in\mathbb{N}}$ of $(x_k)_{k\in\mathbb{N}}$ with $\|y_k\|_G \to 0$. We may thus assume that $\|y_k\|_G < \frac{\varepsilon}{2}$ for all k, where $\varepsilon = \frac{1}{n_4}$. We may construct a block sequence $(z_k)_{k\in\mathbb{N}}$ of $(y_k)_{k\in\mathbb{N}}$ such that $(z_k)_{k\in\mathbb{N}}$ is a $(3, \varepsilon)$ R.I.S. of ℓ_1 averages while each z_k is an average of $(y_k)_{k\in\mathbb{N}}$ with $\|z_k\|_G < \varepsilon$ (see Remark 1.13). Proposition 1.7 yields that the vector $z = \frac{1}{n_4} \sum_{k=1}^{n_4} z_k$ satisfies $\|z\| \leq \frac{2\cdot 3}{m_4} < 1$. On the other hand, the vector z, being a convex combination of $(x_k)_{k\in\mathbb{N}}$, satisfies $\|z\| > 1$. This contradiction completes the proof of the proposition. $\square$

Remark 1.20. The content of the above proposition is that the strictly singular extension (with or without attractors) $X_G = T[G, (\mathcal{A}_{n_j}, \frac{1}{m_j})_j, \sigma]$ of the space Y_G is actually a **reflexive extension**. Namely if $\overline{G}^{w^*}$ is a subset of $c_{00}(\mathbb{N})$ then a consequence of Proposition 1.19 is that the space X_G is reflexive. Furthermore, if X_G is nonreflexive then the quotient space $X_G^*/(X_G)_*$ is norm generated by the classes of the elements of the set $\overline{G}^{w^*}$. Related to this is also the next.

Proposition 1.21. The strictly singular extension (with or without attractors) X_G is reflexively saturated (or somewhat reflexive).

Proof. Let Z be a block subspace of X_G. From the fact that the identity operator $I : X_G \to Y_G$ is strictly singular we may choose a normalized block sequence $(z_n)_{n\in\mathbb{N}}$ in Z, with $\sum\limits_{n=1}^{\infty} \|z_n\|_G < \frac{1}{2}$. We claim that the space $Z' = \overline{\text{span}}\{z_n : n \in \mathbb{N}\}$ is a reflexive subspace of Z.

It is enough to show that the Schauder basis $(z_n)_{n\in\mathbb{N}}$ of Z' is boundedly complete and shrinking. The first follows from the fact that $(z_n)_{n\in\mathbb{N}}$ is a block sequence of the boundedly complete basis $(e_n)_{n\in\mathbb{N}}$ of X_G. To see that $(z_n)_{n\in\mathbb{N}}$ is shrinking it is enough to show that $\|f|_{\overline{\text{span}}\{z_i:\, i\geq n\}}\| \overset{n\to\infty}{\longrightarrow} 0$ for every $f \in X_G^*$. From Proposition 1.19 it is enough to prove it only for $f \in \overline{G}^{w^*}$. Since $\sum\limits_{n=1}^{\infty} \|z_n\|_G < \frac{1}{2}$ the conclusion follows. $\square$

Proposition 1.22. Let Y be an infinite dimensional closed subspace of X_G. Every bounded linear operator $T : Y \to X_G$ takes the form $T = \lambda I_Y + S$ with $\lambda \in \mathbb{R}$ and S a strictly singular operator (I_Y denotes the inclusion map from Y to X_G).

The proof of Proposition 1.22 is similar to the corresponding result for the space of Gowers and Maurey (Lemmas 22 and 23 of [GM1]) and is based on the following lemma.

Lemma 1.23. Let Y be a subspace of X_G and let $T : Y \to X_G$ be a bounded linear operator. Let $(y_l)_{l\in\mathbb{N}}$ be a block sequence of $2-\ell_1^{n_l}$ averages with increasing lengths in Y such that $(Ty_l)_{l\in\mathbb{N}}$ is also a block sequence and $\lim\limits_l \|y_l\|_G = 0$. Then $\lim\limits_l \text{dist}(Ty_l, \mathbb{R}y_l) = 0$.

Proof of Proposition 1.22. Assume that T is not strictly singular. We shall determine a $\lambda \neq 0$ such that $T - \lambda I_Y$ is strictly singular.

Let Y' be an infinite dimensional closed subspace of Y such that $T : Y' \to T(Y')$ is an isomorphism. By standard perturbation arguments and using the fact that X_G is a strictly singular extension of Y_G, we may assume, passing to a subspace, that Y' is a block subspace of X_G spanned by a normalized block sequence $(y_n')_{n\in\mathbb{N}}$ such that $(Ty_n')_{n\in\mathbb{N}}$ is also a block sequence and $\sum\limits_{n=1}^{\infty} \|y_n'\|_G < 1$. From Lemma 1.11 we may choose a block sequence $(y_n)_{n\in\mathbb{N}}$ of $2-\ell_1^{n_i}$ averages of increasing lengths in $\text{span}\{y_n' : n \in \mathbb{N}\}$ with $\|y_n\|_G \to 0$. Lemma

1.23 yields that $\lim_n \mathrm{dist}(Ty_n, \mathbb{R}y_n) = 0$. Thus there exists a $\lambda \neq 0$ such that $\lim_n \|Ty_n - \lambda y_n\| = 0$.

Since the restriction of $T - \lambda I_Y$ to any finite codimensional subspace of $\overline{\mathrm{span}}\{y_n : n \in \mathbb{N}\}$ is clearly not an isomorphism and since also Y is a HI space, it follows from Proposition 1.2 of [AT1] that the operator $T - \lambda I_Y$ is strictly singular. $\square$

2. Strongly strictly singular extensions

It is not known whether the predual $(X_G)_* = \overline{\mathrm{span}}\{e_n^* : n \in \mathbb{N}\}$ of the strictly singular extension X_G is in general a Hereditarily Indecomposable space. In this section we introduce the concept of strongly strictly singular extensions which permit us to ensure the HI property for the space $(X_G)_*$ and to obtain additional information for this space as well as for the spaces $\mathcal{L}(X_G)$, $\mathcal{L}((X_G)_*)$. We also study the quotients of X_G with w^* closed subspaces Z and we show that these quotients are HI.

Definition 2.1. Let G be a ground set and X_G be an extension of the space Y_G. The space X_G is said to be a strongly strictly singular extension provided the following property holds:

For every $C > 0$ there exists $j(C) \in \mathbb{N}$ such that for every $j \geq j(C)$ and every C-bounded block sequence $(x_n)_{n\in\mathbb{N}}$ in X_G with $\|x_n\|_\infty \to 0$ and $(x_n)_{n\in\mathbb{N}}$ a weakly null sequence in Y_G, there exists $L \in [\mathbb{N}]$ such that for every $g \in G$

$$\#\{n \in L : |g(x_n)| > \frac{2}{m_{2j}^2}\} \leq n_{2j-1}.$$

Remark 2.2. Let's observe, for later use, that if $(x_n)_{n\in\mathbb{N}}$ is a R.I.S. of ℓ_1 averages (Remark 1.13), then $\|x_n\|_\infty \to 0$. Therefore if X_G is a strongly singular extension of Y_G, there exists a subsequence $(x_{l_n})_{n\in\mathbb{N}}$ such that the sequence $y_n = x_{l_{2n-1}} - x_{l_{2n}}$ is weakly null and satisfies the above stated property.

Proposition 2.3. If X_G is a strongly strictly singular extension (with or without attractors) of Y_G, then the identity map $I : X_G \to Y_G$ is a strictly singular operator.

Proof. In any block subspace of X_G we may consider a block sequence $(x_n)_{n\in\mathbb{N}}$ with $1 \leq \|x_n\|_{X_G} \leq 2$, $\|x_n\|_\infty \to 0$ and $(x_n)_{n\in\mathbb{N}}$ being weakly null. Passing to a subsequence $(x_{l_n})_{n\in\mathbb{N}}$ for $j \geq j(2)$ we obtain that

$$\|\frac{1}{n_{2j}} \sum_{i=1}^{n_{2j}} x_{l_i}\|_{X_G} \geq \frac{1}{m_{2j}}$$

and on the other hand

$$\|\frac{1}{n_{2j}} \sum_{i=1}^{n_{2j}} x_{l_i}\|_{Y_G} \leq \frac{2}{m_{2j}^2} + \frac{2n_{2j-1}}{n_{2j}} < \frac{3}{m_{2j}^2}$$

which yields that I is not an isomorphism in any block subspace of X_G. $\square$

Definition 2.1 is the analogue of the definition of $\mathcal{S}_2$ bounded or $\mathcal{S}_\xi$ bounded sets (see [AT1] where the norming sets are defined with the use of saturation methods of the form $(\mathcal{S}_{\xi_j}, \frac{1}{m_j})_j)$ in the context of saturation methods of low complexity, i.e. of the form $(\mathcal{A}_{n_j}, \frac{1}{m_j})_j$. As we have noticed earlier the assumption of a strongly strictly singular extension (with or without attractors) is required in order to prove that the predual space $(X_G)_*$ is Hereditarily Indecomposable.

The HI property of the dual space X_G^* essentially depends on the internal structure of the set G. Thus we shall see examples of strongly strictly singular extensions (with or without attractors) such that X_G^* is either HI or contains $\ell_2(\mathbb{N})$.

Definition 2.4. **(c_0^k vectors)** Let $k \in \mathbb{N}$. A finitely supported vector $x^* \in (X_G)_*$ is said to be a $C - c_0^k$ vector if there exist $x_1^* < \cdots < x_k^*$ such that $\|x_i^*\| > C^{-1}$, $x^* = x_1^* + \cdots + x_k^*$ and $\|x^*\| \leq 1$.

Remark 2.5. The fact that the norming set D_G is rationally convex yields that D_G is pointwise dense in the unit ball of the space $B_{X_G^*}$. Since also the norming set D_G is closed in $(\mathcal{A}_{n_{2j}}, \frac{1}{m_{2j}})$ operations we get that for every j, if $f_1, f_2, \ldots, f_{n_{2j}}$ is a block sequence in X_G^* with $\|f_i\| \leq 1$ then $\|\frac{1}{m_{2j}} \sum_{i=1}^{n_{2j}} f_i\| \leq 1$.

Lemma 2.6. Let $(x_\ell^*)_{\ell \in \mathbb{N}}$ be a block sequence in $(X_G)_*$. Then for every k there exists a $x^* \in \operatorname{span}\{x_\ell^* : \ell \in \mathbb{N}\}$ which is a $2 - c_0^k$ vector.

The proof is based on Remark 2.5 and can be found in [AT2] (Lemma 5.4).

Lemma 2.7. For every $2 - c_0^k$ vector x^* and every $\varepsilon > 0$ there exists a $2 - c_0^k$ vector f with $f \in D_G$, $\operatorname{ran} f = \operatorname{ran} x^*$ and $\|x^* - f\| < \varepsilon$.

Proof. This follows from the fact that the norming set D_G is pointwise dense in $B_{X_G^*}$. □

Lemma 2.8. If x^* is a $C - c_0^k$ vector then there exists a $C - \ell_1^k$ average x with $\operatorname{ran}(x) = \operatorname{ran}(x^*)$ and $x^*(x) > 1$.

Proof. Let $x^* = \sum_{i=1}^{k} x_i^*$ where $x_1^* < \cdots < x_k^*$, $\|x_i^*\| > C^{-1}$ and $\|x^*\| \leq 1$. For $i = 1, \ldots, k$ we choose $x_i \in X_G$ with $\|x_i\| \leq 1$, $x_i^*(x_i) > C^{-1}$ and $\operatorname{ran}(x_i) = \operatorname{ran}(x_i^*)$. We set $x = \frac{1}{k} \sum_{i=1}^{k} (Cx_i)$. Then $\|Cx_i\| \leq C$ for $i = 1, \ldots, k$, while $\|x\| \geq x^*(x) > 1$. Also, since $\operatorname{ran}(x) = \operatorname{ran}(x^*)$, x is the desired $C - \ell_1^k$ average. □

Proposition 2.9. Let Z be a block subspace of $(X_G)_*$ and let $k \in \mathbb{N}$, $\varepsilon > 0$. Then there exists a $2 - \ell_1^k$ vector y and $y^* \in D_G$ such that $y^*(y) > 1$, $\operatorname{ran}(y^*) = \operatorname{ran}(y)$ and $\operatorname{dist}(y^*, Z) < \varepsilon$.

Proof. From Lemma 2.6 we can choose a $2 - c_0^k$ vector $x^* = \sum_{i=1}^{k} x_i^*$ in Z. Lemma 2.8 yields the existence of a $2 - \ell_1^k$ average y with $\operatorname{ran}(y) = \operatorname{ran}(x^*)$ and

$x^*(y) > 1$. Applying Lemma 2.7 we can find $y^* \in D_G$ with $\mathrm{ran}(y^*) = \mathrm{ran}(x^*)$ and $\|y^* - x^*\| < \min\{\varepsilon, \frac{x^*(y)-1}{2}\}$. It is clear that y and y^* satisfy the desired conditions. □

Lemma 2.10. Suppose that X_G is a strongly strictly singular extension (with or without attractors) and let Z be a block subspace of $(X_G)_*$. Then for every $j > 1$ and $\varepsilon > 0$ there exists a $(18, 2j, 1)$ exact pair (z, z^*) with $\mathrm{dist}(z^*, Z) < \varepsilon$ and $\|z\|_G \leq \frac{3}{m_{2j}}$.

Proof. Using Proposition 2.9 we may select a block sequence $(y_l)_{l\in\mathbb{N}}$ in X_G and a sequence $(y_l^*)_{l\in\mathbb{N}}$ such that

(i) Each y_l is a $2 - \ell_1^{n_{i_l}}$ average where $(i_l)_{l\in\mathbb{N}}$ is an increasing sequence of integers.

(ii) $y_l^* \in D_G$ for all l and $\sum_{l=1}^{\infty} \mathrm{dist}(y_l^*, Z) < \varepsilon$.

(iii) $y_l^*(y_l) > 1$ and $\mathrm{ran}\, y_l^* = \mathrm{ran}\, y_l$.

From Remark 1.13 we may assume (passing, if necessary, to a subsequence) that $(y_l)_{l\in\mathbb{N}}$ is a $(3, \varepsilon)$ R.I.S. Since Y_G contains no isomorphic copy of ℓ_1 we may assume, passing again to a subsequence, that $(y_l)_{l\in\mathbb{N}}$ is a weakly Cauchy sequence in Y_G. Setting $x_l = \frac{1}{2}(y_{2l-1} - y_{2l})$ it is clear that $\|x_l\|_\infty \to 0$ while $(x_l)_{l\in\mathbb{N}}$ is a weakly null sequence in Y_G. From the fact that X_G is a strongly strictly singular extension (with or without attractors) it follows that there exists $M \in [\mathbb{N}]$ such that for every $g \in G$ the set $\{l \in M : \ |g(x_l)| > \frac{2}{m_{2j}^2}\}$ has at most n_{2j-1} elements (notice that $66 m_{2j}^4 < n_{2j-1}$). We may assume that $M = \mathbb{N}$. Also $(x_l)_{l\in\mathbb{N}}$ is a $(3, \varepsilon)$ R.I.S. We set

$$z^* = \frac{1}{m_{2j}} \Big(\sum_{l=1}^{n_{2j}-1} y_{2l-1}^* - y_{2l}^* \Big).$$

From Proposition 1.7 we get that $\|\frac{1}{n_{2j}} \sum_{l=1}^{n_{2j}} x_l\| \leq \frac{6}{m_{2j}}$ while $z^*(\frac{m_{2j}}{n_{2j}} \sum_{l=1}^{n_{2j}} x_l) > 1$ hence there exists η with $\frac{1}{6} \leq \eta < 1$ such that $z^*(\eta \frac{m_{2j}}{n_{2j}} \sum_{l=1}^{n_{2j}} x_l) = 1$. We set

$$z = \eta \frac{m_{2j}}{n_{2j}} \sum_{l=1}^{n_{2j}} x_l.$$

It follows easily from Proposition 1.7 that (z, z^*) is a $(18, 2j, 1)$ exact pair. From condition (ii) we get that $\mathrm{dist}(z^*, Z) < \varepsilon$. Finally we have that $\|z\|_G \leq \frac{3}{m_{2j}}$. Indeed, let $g \in G$. Since $\#\{l : \ |g(x_l)| > \frac{2}{m_{2j}^2}\} \leq n_{2j-1}$ and $|g(x_l)| \leq \|x_l\| \leq 2$ for all l we have that

$$|g(z)| \leq \frac{m_{2j}}{n_{2j}} \sum_{l=1}^{n_{2j}} |g(x_l)| \leq \frac{m_{2j}}{n_{2j}} \Big(\frac{2}{m_{2j}^2} n_{2j} + 2 n_{2j-1}\Big) < \frac{3}{m_{2j}}.$$

Therefore $\|z\|_G \leq \frac{3}{m_{2j}}$. □

Lemma 2.11. Let X_G be a strongly strictly singular extension (with or without attractors) and let Y, Z be a pair of block subspaces of $(X_G)_*$. Then for every $\varepsilon > 0$ and $j > 1$ there exists a $(18, 4j-1, 1)$ dependent sequence $(x_k, x_k^*)_{k=1}^{n_{4j-1}}$ with $\sum \operatorname{dist}(x_{2k-1}^*, Y) < \varepsilon$, $\sum \operatorname{dist}(x_{2k}^*, Z) < \varepsilon$ and $\|x_k\|_G \leq \frac{2}{n_{4j-1}^2}$ for all k.

Proof. This is an immediate consequence of Lemma 2.10. □

Theorem 2.12. If X_G is a strongly strictly singular extension (with or without attractors), then the predual space $(X_G)_*$ is Hereditarily Indecomposable.

Proof. Let Y, Z be a pair of block subspaces of $(X_G)_*$. For every $j > 1$ using Lemma 2.11 we may select a $(18, 4j-1, 1)$ dependent sequence $(x_k, x_k^*)_{k=1}^{n_{4j-1}}$ with $\sum \operatorname{dist}(x_{2k-1}^*, Y) < 1$ and $\sum \operatorname{dist}(x_{2k}^*, Z) < 1$ and $\|x_k\|_G \leq \frac{2}{m_{4j-1}^2}$ for all k.

The functional $x^* = \frac{1}{m_{4j-1}} \sum_{k=1}^{n_{4j-1}} x_k^*$ belongs to the norming set D_G hence $\|x^*\| \leq 1$. From Proposition 1.17 we get that $\|\frac{1}{n_{4j-1}} \sum_{k=1}^{n_{4j-1}} (-1)^{k+1} x_k\| \leq \frac{144}{m_{4j-1}^2}$.

We set

$$h_Y = \frac{1}{m_{4j-1}} \sum_{k=1}^{n_{4j-1}/2} x_{2k-1}^* \text{ and } h_Z = \frac{1}{m_{4j-1}} \sum_{k=1}^{n_{4j-1}/2} x_{2k}^*.$$

Estimating $h_Y - h_Z$ on the vector $\frac{1}{n_{4j-1}} \sum_{k=1}^{n_{4j-1}} (-1)^{k+1} x_k$ yields that $\|h_Y - h_Z\| \geq \frac{m_{4j-1}}{144}$ while we obviously have that $\|h_Y + h_Z\| = \|x^*\| \leq 1$.

From the fact that $\operatorname{dist}(h_Y, Y) < 1$ and $\operatorname{dist}(h_Z, Z) < 1$ we may select $f_Y \in Y$ and $f_Z \in Z$ with $\|h_Y - f_Y\| < 1$ and $\|h_Z - f_Z\| < 1$. From the above estimates we conclude that $\|f_Y - f_Z\| \geq (\frac{m_{4j-1}}{432} - \frac{2}{3})\|f_Y + f_Z\|$. Since we can find such f_Y and f_Z for arbitrary large j it follows that $(X_G)_*$ is Hereditarily Indecomposable. □

The next two theorems concern the structure of $\mathcal{L}(\mathfrak{X}_G)$, $\mathcal{L}((\mathfrak{X}_G)_*)$. We start with the following lemmas. The first is the analogue of Lemma 1.23 for strongly strictly singular extensions.

Lemma 2.13. Assume that X_G is a strongly strictly singular extension. Let Y be a subspace of X_G and let $T : Y \to X_G$ be a bounded linear operator. Let $(y_\ell)_{\ell \in \mathbb{N}}$ be a block sequence in Y of C-$\ell_1^{j_k}$ averages with $\lim j_k = \infty$. Furthermore assume that $(Ty_\ell)_{\ell \in \mathbb{N}}$ is also a block sequence. Then

$$\lim \operatorname{dist}(Ty_\ell, \mathbb{R}y_\ell) = 0.$$

Proof. Assume that the conclusion fails. We may assume, passing to a subsequence that there exists $\delta > 0$ such that for every $\ell \in \mathbb{N}$, $\operatorname{dist}(Ty_\ell, \mathbb{R}y_\ell) > \delta$ and moreover that $(y_\ell)_\ell$ is a R.I.S. Next for each $\ell \in \mathbb{N}$, we choose ϕ_ℓ such that $\operatorname{supp} \phi_\ell \subset \operatorname{ran}(y_\ell \cup Ty_\ell)$, $\phi_\ell \in D_G$, $\phi_\ell(Ty_\ell) > \frac{\delta}{2}$ and $\phi_\ell(y_\ell) = 0$. From Remark 2.2 and since X_G is a strictly singular extension of Y_G, for every

$j \in \mathbb{N}$, $j > j(C)$ we can find a subsequence $(y_{\ell_k})_{k\in\mathbb{N}}$ such that the sequence $w_k = (y_{\ell_{2k-1}} - y_{\ell_{2k}})/2$ is weakly null and for every $g \in G$,

$$\#\Big\{k \in \mathbb{N} : |g(w_k)| > \frac{2}{m_{2j}^2}\Big\} \leq n_{2j-1}.$$

This yields that for every $j > j(C)$ there exists $w_{k_1} < w_{k_2} < \cdots < w_{k_{n_{2j}}}$ and $\phi_{k_1} < \phi_{k_2} < \cdots < \phi_{k_{n_{2j}}}$ such that setting $w\frac{m_{2j}}{n_{2j}} \sum_{i=1}^{n_{2j}} w_{k_i}$ and $\phi\frac{1}{m_{2j}} \sum_{i=1}^{n_{2j}} \phi_i$, we have that

$$\|w\| \leq 6C, \quad \phi \in D_G, \quad \phi(Tw) > \frac{\delta}{2} \quad \phi(w) = 0, \quad \text{and} \quad \|w\|_G < \frac{3}{m_{2j}^2}.$$

In particular (w, ϕ) is $(6C, 2j, 0)$ exact pair with $\|w\|_G < \frac{3}{m_{2j}^2}$. The remaining part of the proof follows the arguments of Lemmas 22 and 23 of [GM1] using Proposition 1.17, (iii). $\square$

The next lemma is easy and its proof is included in the proof of Theorem 9.4 of [AT1].

Lemma 2.14. Let X be a Banach space with a boundedly complete basis $(e_n)_{n\in\mathbb{N}}$ not containing ℓ_1. Assume that $T : X \to X$ is a bounded linear non weakly compact operator. Then there exist two block sequences $(x_n)_{n\in\mathbb{N}}$ $(y_n)_{n\in\mathbb{N}}$ and y in X such that the following hold:

(i) $(x_n)_{n\in\mathbb{N}}$ is normalized, $x_n \overset{w^*}{\to} x^{**} \in X^{**} \setminus X$.
(ii) $(y_n)_{n\in\mathbb{N}}$ is bounded, $y_n \overset{w^*}{\to} y^{**} \in X^{**} \setminus X$.
(iii) $\|Tx_n - (y + y_n)\| \to 0$.

Theorem 2.15. If X_G is a strongly strictly singular extension (with or without attractors), then every bounded linear operator $T : X_G \to X_G$ takes the form $T = \lambda I + S$ with S a strictly singular and weakly compact operator.

Proof. We already know from Proposition 1.22 that every bounded linear operator $T : X_G \to X_G$ is of the form $T = \lambda I + S$ with S a strictly singular operator so it remains to show that every strictly singular operator $S : X_G \to X_G$ is weakly compact.

Assume now that there exists a strictly singular $T \in \mathcal{L}(X_G)$ which is not weakly compact. Then from Lemma 2.14, there exist $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$, y in X_G satisfying the conclusions of Lemma 2.14. It follows that there exists a subsequence $(x_n)_{n\in L}$ such that setting $Z = \overline{\text{span}}\{x_n : \ n \in L\}$ there exists a compact perturbation of $T|Z$ denoted by $\tilde{T}$ such that for $n \in L$, we have that $\tilde{T}(x_n) = y + z_n$. For simplicity of notation assume that $L = \mathbb{N}$. Since $x_n \overset{w^*}{\to} x^{**} \in X_G^{**} \setminus X_G$ and $y_n \overset{w^*}{\to} y^{**} \in X_G^{**} \setminus X_G$ we may assume that every convex combination of $(x_n)_{n\in\mathbb{N}}$ has norm greater than $\delta > 0$.

Choose $(z_k)_{k\in\mathbb{N}}$, $z_k = \frac{1}{n_{j_k}} \sum_{i\in F_k} \frac{1}{\delta} x_i$ with $\#F_k = n_{j_k}$ and $F_k < F_{k+1}$. Then setting $w_k = \frac{z_{2k-1} - z_{2k}}{\|z_{2k-1} - z_{2k}\|}$, Lemma 2.13 (actually its proof) yields that

$$\lim \text{dist}(\tilde{T}w_k, \mathbb{R}w_k) = 0.$$

From this we conclude that for some subsequence $(w_k)_{k\in L}$, $\tilde{T}|\overline{\mathrm{span}}\{w_k:\ k\in L\}$ is an isomorphism contradicting our assumption that T is strictly singular. $\square$

Theorem 2.16. Let X_G be a strongly singular extension (with or without attractors) of Y_G. Then every bounded linear operator $T:(X_G)_*\to(X_G)_*$ is of the form $T=\lambda I+S$ with S strictly singular.

The proof of this result follows the lines of Proposition 7.1 in [AT2]. We first state two auxiliary lemmas.

Lemma 2.17. Let X be a HI space with a Schauder basis $(e_n)_{n\in\mathbb{N}}$. Assume that $T:X\to X$ is a bounded linear operator which is not of the form $T=\lambda I+S$ with S strictly singular. Then there exists n_0 and $\delta>0$ such that for every $z\in X_{n_0}=\overline{\mathrm{span}}\{e_n:\ n\geq n_0\}$, $\mathrm{dist}(Tz,\mathbb{R}z)\geq\delta\|z\|$.

Proof. If not, then there exists a normalized block sequence $(z_n)_{n\in\mathbb{N}}$ such that $\mathrm{dist}(Tz_n,\mathbb{R}z_n)\leq\frac{1}{n}$. Choose $\lambda\in\mathbb{R}$ such that $\|Tz_n-\lambda z_n\|_{n\in L}\to 0$ for a subsequence $(z_n)_{n\in L}$. Then for a further subsequence $(z_n)_{n\in M}$ we have that $T-\lambda I|_{\overline{\mathrm{span}}\{z_n:\ n\in M\}}$ is a compact operator. The HI property of X easily yields that $T-\lambda I$ is a strictly singular operator, contradicting our assumption. $\square$

Lemma 2.18. Let $T:(X_G)_*\to(X_G)_*$ be a bounded linear operator with $\|T\|=1$. Assume that for some $\delta>0$, and $n_0\in\mathbb{N}$, $\mathrm{dist}(Tf,\mathbb{R}f)\geq\delta\|f\|$ for all $f\in(X_G)_*$ with $n_0<\mathrm{supp}\,f$. Then for every $k\in\mathbb{N}$ and every block subspace Z of $(X_G)_*$ there exist a $z^*\in Z$ with $\|z^*\|\leq 1$ and a $\frac{2}{\delta}$-ℓ_1^k average z such that $z^*(z)=0$, $Tz^*(z)>1$ and $\mathrm{ran}\,z\subset\mathrm{ran}\,z^*\cup\mathrm{ran}\,Tz^*$.

Proof. By Lemma 2.6 there exists a 2-c_0^k vector $z^*=\sum\limits_{i=1}^{k}z_i^*$ in Z with $n_0<\min\mathrm{supp}\,z^*$. Since $\mathrm{dist}(Tz_i^*,\mathbb{R}z_i^*)\geq\delta\|z_i^*\|>\frac{\delta}{2}$ we may choose for each $i=1,\dots,k$ a vector $z_i\in X_G$ with $\|z_i\|<\frac{2}{\delta}$ and $\mathrm{supp}\,z_i\subset\mathrm{ran}\,z_i^*\cup\mathrm{ran}\,Tz_i^*$ satisfying $Tz_i^*(z_i)>1$ and $z_i^*(z_i)=0$. We set $z=\frac{1}{k}\sum\limits_{i=1}^{k}z_i$. It is easy to check that z is the desired vector. $\square$

Proof of Theorem 2.16. On the contrary assume that there exists $T\in\mathcal{L}((X_G)_*)$ which is not of the desired form. Assume further that $\|T\|=1$ and Te_n^* is finitely supported with $\lim\min\mathrm{supp}\,Te_n^*=\infty$. (We may assume the later conditions from the fact that the basis $(e_n^*)_{n\in\mathbb{N}}$ of $(X_G)_*$ is weakly null.) In particular for every block sequence $(z_n^*)_{n\in\mathbb{N}}$ in $(X_G)_*$ there exists a subsequence $(z_n^*)_{n\in L}$ such that $(\mathrm{ran}\,z_n^*\cup\mathrm{ran}\,Tz_n^*)_n$ is a sequence of successive subsets of $\mathbb{N}$.

Let $\delta>0$ and $n_0\in\mathbb{N}$ be as in Lemma 2.18 and let $j(\frac{2}{\delta})$ be the corresponding index such that for all $j\geq j(\frac{2}{\delta})$ the conclusion of Definition 2.1 holds for $\frac{2}{\delta}$-bounded block sequences of X_G. Using arguments similar to those of Lemma 2.10 for $j\geq j(\frac{2}{\delta})$ we can find an $(\frac{18}{\delta},2j,0)$ exact pair (z,z^*), with $Tz^*(z)>1$ and $\|z\|_G\leq\frac{3}{m_{2j}}$. Then for every $j\in\mathbb{N}$ there exists a $(\frac{18}{\delta},4j-1,0)$ dependent sequence $(z_k,z_k^*)_{k=1}^{n_{4j-1}}$, such that $z_k^*(z_k)=0$, $Tz_k^*(z_k)>1$, $\|z_k\|_G\leq\frac{1}{m_{4j-1}^2}$,

$(\operatorname{ran} z_k^* \cup \operatorname{ran} Tz_k^*)_{k=1}^{n_{4j-1}}$ are successive subsets of $\mathbb{N}$ and $\operatorname{ran} z_k \subset I_k$ where I_k is the minimal interval of $\mathbb{N}$ containing $\operatorname{ran} z_k^* \cup \operatorname{ran} Tz_k^*$.

Proposition 1.17 yields that

$$\|\frac{1}{n_{4j-1}} \sum_{k=1}^{n_{4j-1}} z_k\| \leq \frac{144}{m_{4j-1}^2 \delta}.$$

Finally $\|\frac{1}{m_{4j-1}} \sum_{k=1}^{n_{4j-1}} Tz_k^*\| \leq 1$ (since $\|T\| \leq 1$) and also

$$1 \geq \|\frac{1}{m_{4j-1}} \sum_{k=1}^{n_{4j-1}} Tz_k^*\| \geq \frac{m_{4j-1}^2 \delta}{144 m_{4j-1}} \frac{1}{n_{4j-1}} \sum_{k=1}^{n_{4j-1}} Tz_k^*(z_k) \geq \frac{m_{4j-1}\delta}{144}.$$

This yields a contradiction for sufficiently large $j \in \mathbb{N}$. $\square$

The following lemma is similar to a corresponding result used by V. Ferenczi [Fe] in order to show that every quotient of the space constructed by W.T Gowers and B. Maurey remains Hereditarily Indecomposable.

Lemma 2.19. Suppose that X_G is a strictly singular extension of Y_G (with or without attractors). Let Z be w^* closed subspace of X_G and let Y be a closed subspace of X_G with $Z \subset Y$ such that the quotient space Y/Z is infinite dimensional. Then for every $m, N \in \mathbb{N}$ and $\varepsilon > 0$ there exists $x \in \operatorname{span}\{e_i : i \geq m\}$ which is a $2 - \ell_1^N$ average with $\operatorname{dist}(x, Y) < \varepsilon$ and there exists $f \in B_{(X_G)_*}$ with $\operatorname{dist}(f, Z_\perp) < \varepsilon$ such that $f(x) > 1$.

Proof. We recall that from the fact that Z is w^* closed the quotient space X_G/Z may be identified with the dual of the annihilator $Z_\perp = \{f \in (X_G)_* : f(z) = 0 \quad \forall z \in Z\}$, i.e. $X_G/Z = (Z_\perp)^*$. Pick a normalized sequence $(\hat{y}'_n)_{n\in\mathbb{N}}$ in Y/Z with $\hat{y}'_n \overset{w^*}{\to} 0$. From W.B. Johnson's and H.P. Rosenthal's work on w^*- basic sequences ([JR]) and their w^* analogue of the classical Bessaga - Pelczynski theorem, we may assume, passing to a subsequence, that $(\hat{y}'_n)_{n\in\mathbb{N}}$ is a w^* basic sequence. Hence there exists a bounded sequence $(z_n^*)_{n\in\mathbb{N}}$ in $Z_\perp$ such that $(z_n^*, \hat{y}'_n)_{n\in\mathbb{N}}$ are biorthogonal ($z_n^*(\hat{y}'_m) = \delta_{nm}$) and $\sum_{i=1}^{n} \hat{y}(z_n^*)\hat{y}'_i \to \hat{y}$ for every $\hat{y}$ in the weak* closure of the linear span of the sequence $(\hat{y}'_n)_{n\in\mathbb{N}}$.

Since $(X_G)_*$ contains no isomorphic copy of ℓ_1 (as it a space with separable dual) we may assume, passing to a subsequence, that $(z_n^*)_{n\in\mathbb{N}}$ is weakly Cauchy, hence $(z_{2n-1}^* - z_{2n}^*)_{n\in\mathbb{N}}$ is weakly null. Using a sliding hump argument and passing to a subsequence we may assume that with an error up to ε this sequence is a block sequence with respect to the standard basis of $(X_G)_*$.

We set $y_n^* = z_{2n-1}^* - z_{2n}^*$ and $\hat{y}_n = \hat{y}'_{2n-1}$ for $n = 1, 2, \ldots$. Then $(y_n^*, \hat{y}_n)_{n\in\mathbb{N}}$ are biorthogonal, $(y_n^*)_{n\in\mathbb{N}}$ is a weakly null block sequence in $(X_G)_*$ with $y_n^* \in Z_\perp$, while $(\hat{y_n})$ is a normalized w^*- basic sequence in Y/Z.

We choose $k, j \in \mathbb{N}$ such that $2^k > m_{2j}$ and $(2N)^k \leq n_{2j}$. We set

$$\mathcal{A}_1 = \{L \in [\mathbb{N}], \;\; L = \{l_i, \; i \in \mathbb{N}\} : \; \|\frac{1}{2N} \sum_{i=1}^{2N} (-1)^{i+1} \hat{y}_{l_i}\| > \frac{1}{2}\}$$

$$\text{and} \qquad \mathcal{B}_1 = [\mathbb{N}] \setminus \mathcal{A}_1$$

From Ramsey's theorem we may find a homogenous set L either in $\mathcal{A}_1$ or in $\mathcal{B}_1$. We may assume that $L = \mathbb{N}$.

Suppose first that the homogenous set is in $\mathcal{A}_1$, i.e. $\|\frac{1}{2N}\sum_{i=1}^{2N}(-1)^{i+1}\hat{y}_{l_i}\| > \frac{1}{2}$ for every $l_1 < l_2 < \ldots < l_{2N}$ in $\mathbb{N}$. For each n we may choose $y_n \in Y \subset X_G$ with $\|y_n\| = 1$ and $Q(y_n) = \hat{y}_n$. Passing to a subsequence we may assume (again with an error up to ε) that the sequence $x_n = y_{2n-1} - y_{2n}$ is a weakly null block sequence in X_G with $\min\operatorname{supp} x_i \geq m$. We set $x = \frac{1}{N}\sum_{i=1}^{N} x_i$. It is clear that x is a 2-ℓ_1^N average while since $Qx \in Y/Z \subset X_G/Z = (Z_\perp)^*$ and $\|Qx\| > 1$ there exists $f \in Z_\perp$ with $\|f\| \leq 1$ such that $f(x) > 1$.

On the other hand if the homogenous set is in $\mathcal{B}_1$ then we may assume, passing again to a subsequence that there exists $a_1 \geq 2$ such that setting $\hat{y}_{2,n} = a_1 \cdot \frac{1}{2N}\sum_{i=(n-1)(2N)+1}^{n(2N)}(-1)^{i+1}\hat{y}_i$ for $i = 1, 2, \ldots$, $(\hat{y}_{2,n})_{n\in\mathbb{N}}$ is a normalized sequence in Y/Z. We may again apply Ramsey's theorem defining $\mathcal{A}_2$, $\mathcal{B}_2$ as before, using the sequence $(\hat{y}_{2,n})_{n\in\mathbb{N}}$ instead of $(\hat{y}_n)_{n\in\mathbb{N}}$. If the homogenous set is in $\mathcal{A}_2$ the proof finishes as before while if it is in $\mathcal{B}_2$ we continue defining $(\hat{y}_{3,n})_{n\in\mathbb{N}}$ $\mathcal{A}_3$, $\mathcal{B}_3$ and so on.

If in none of the first k steps we arrived at a homogenous set in some $\mathcal{A}_i$ then there exist $a_1, a_2, \ldots, a_k \geq 2$ and $l_1 < l_2 < \cdots < l_{(2N)^k}$ in $\mathbb{N}$ such that the vector

$$\hat{y} = a_1 a_2 \cdot \ldots \cdot a_k \frac{1}{(2N)^k}\sum_{i=1}^{(2N)^k}(-1)^{i+1}\hat{y}_{l_i}$$

satisfies $\|\hat{y}\| = 1$.

But then the functional $y^* = \frac{1}{m_{2j}}\sum_{i=1}^{(2N)^k}(-1)^{i+1}y^*_{l_i}$ belongs to $Z_\perp$ and satisfies $\|y^*\| \leq 1$ (as $(y^*_{l_i})_i$ is a block sequence with $\|y^*_{l_i}\| \leq 1$ and $(2N)^k \leq n_{2j}$). Therefore, taking into account the biorthogonality, we ge that

$$1 = \|\hat{y}\| \leq y^*(\hat{y}) = \frac{1}{m_{2j}}\frac{2^k}{(2N)^k}(2N)^k = \frac{2^k}{m_{2j}}$$

which contradicts our choice of k and j. □

Lemma 2.20. Suppose that X_G is a strongly strictly singular extension of Y_G (with or without attractors) and let Y and Z be as in Lemma 2.19. Then for every $j > 1$ and every $\varepsilon > 0$ there exists a $(18, 2j, 1)$ exact pair (y, f) with $\operatorname{dist}(y, Y) < \varepsilon$, $\|y\|_G < \frac{3}{m_{2j}}$ and $\operatorname{dist}(f, Z_\perp) < \varepsilon$.

Proof. Let $(\varepsilon_i)_{i\in\mathbb{N}}$ be a sequence of positive reals with $\sum_{i=1}^{\infty}\varepsilon_i < \varepsilon$. Using Lemma 2.19 we may inductively construct a block sequence $(x_i)_{i\in\mathbb{N}}$ in X_G, a sequence $(\phi_i)_{i\in\mathbb{N}}$ in $B_{(X_G)_*}$ and a sequence of integers $t_1 < t_2 < \cdots$ such that the following are satisfied:

(i) $\operatorname{dist}(x_i, Y) < \varepsilon_i$, $\operatorname{dist}(\phi_i, Z_\perp) < \varepsilon_i$ and $\phi_i(x_i) > 1$.

(ii) The sequence $(x_i)_{i\in\mathbb{N}}$ is a sequence of $2-\ell_1$ averages of increasing length and $\min\operatorname{supp} x_i \geq t_i$.

(iii) The restriction of the functional ϕ_i to the space $\overline{\operatorname{span}}\{e_n : n \geq t_{i+1}\}$ has norm at most ε_i.

Passing to a subsequence we may assume that the sequences $(\phi_i)_{i\in\mathbb{N}}$ and $(x_i)_{i\in\mathbb{N}}$ are weakly Cauchy. Thus the sequence $(-\phi_{2n-1}+\phi_{2n})_{n\in\mathbb{N}}$ is weakly null; so we may assume, passing again to a subsequence, that it is a block sequence and that, since $(x_{2n})_{n\in\mathbb{N}}$ is a weakly Cauchy sequence, the sequence $y_n = x_{4n-2} - x_{4n}$ is a weakly null block sequence in X_G and thus also in Y_G, therefore, from the fact that X_G is a strongly strictly singular extension of Y_G we may assume, passing to a subsequence, that for every $g \in G$ the set $\{n \in \mathbb{N} : \ |g(y_n)| > \frac{2}{m_{2j}^2}\}$ contains at most n_{2j-1} elements and also that $(y_n)_{n\in\mathbb{N}}$ is a $(3,\varepsilon)$ R.I.S. of ℓ_1 averages.

We set $f_n = \frac{1}{2}(-\phi_{4n-3}+\phi_{4n-2})$ for $n = 1, 2, \ldots$, and we may assume that $\max(\operatorname{supp} f_n \cup \operatorname{supp} y_n) < \min(\operatorname{supp} f_{n+1} \cup \operatorname{supp} y_{n+1})$ for all n. Finally we set

$$y' = \frac{m_{2j}}{n_{2j}} \sum_{i=1}^{n_{2j}} y_i \text{ and } f = \frac{1}{m_{2j}} \sum_{i=1}^{n_{2j}} f_i.$$

As in the proof of Lemma 2.10 we obtain that (y, f), where y is a suitable scalar multiple of y', is the desired exact pair. □

Using Lemma 2.20 we prove the following:

Theorem 2.21. If X_G is a strongly strictly singular extension of Y_G (with or without attractors) and Z is a w^* closed subspace of X_G of infinite codimension then the quotient space X_G/Z is Hereditarily Indecomposable.

Proof. Let Y_1 and Y_2 be subspaces of X_G with $Z \hookrightarrow Y_1 \cap Y_2$ such that Z is of infinite codimension in each Y_i, $i = 1, 2$. Then for every $\varepsilon > 0$ and $j \in \mathbb{N}$ we may select an $(18, 4j-1, 1)$ dependent sequence $\chi = (x_k, x_k^*)_{k=1}^{n_{4j-1}}$ such that

(i) $\|x_k\|_G < \frac{1}{m_{4j-1}^2}$, $k = 1, \ldots, n_{4j-1}$.

(ii) $\operatorname{dist}(x_{2k-1}, Y_1) < \varepsilon$, $\operatorname{dist}(x_{2k}, Y_2) < \varepsilon$.

(iii) $\operatorname{dist}(x_k^*, Z_\perp) < \varepsilon$.

Let $Q : X_G \to X_G/Z$ be the quotient map. If $\varepsilon > 0$ is sufficiently small Proposition 1.17 easily yields that

$$\operatorname{dist}(S_{Q(Y_1)}, S_{Q(Y_2)}) < \frac{C}{m_{4j-1}}$$

where C is a constant independent of j. The proof is complete. □

3. The James tree space $JT_{\mathcal{F}_2}$.

In this section we define a class of James Tree-like spaces. These spaces share some of the main properties of the classical JT space. Namely they do not contain an isomorphic copy of the space ℓ_1. Furthermore they have a bimonotone basis. In particular their norming set is a ground set and a specific example of this form will be the ground set for our final constructions. The principal goal is to prove the inequality in Proposition 3.14 yielding that

that the ground set $\mathcal{F}_2$ defined in the next section admits a strongly strictly singular extension. In Appendix B we present a systematic study of $JT_{\mathcal{F}_2}$ spaces and of some variants of them.

Definition 3.1. (JTG families) A family $\mathcal{F} = (F_j)_{j=0}^{\infty}$ of subsets of $c_{00}(\mathbb{N})$ is said to be a **James Tree Generating family** (JTG family) provided it satisfies the following conditions:

(A) $F_0 = \{\pm e_n^* : n \in \mathbb{N}\}$ and each F_j is nonempty, countable, symmetric, compact in the topology of pointwise convergence and closed under restrictions to intervals of $\mathbb{N}$.

(B) Setting $\tau_j = \sup\{\|f\|_\infty : f \in F_j\}$, the sequence $(\tau_j)_{j\in\mathbb{N}}$ is strictly decreasing and $\sum\limits_{j=1}^{\infty} \tau_j \le 1$.

Definition 3.2. (The $\sigma_{\mathcal{F}}$ coding) Let $(F_j)_{j=0}^{\infty}$ be a JTG family. We fix a pair Ξ_1, Ξ_2 of disjoint infinite subsets of $\mathbb{N}$. Let $W = \{(f_1, \dots, f_d) : f_i \in \cup_{j=1}^{\infty} F_j,\ f_1 < \dots < f_d,\ d \in \mathbb{N}\}$. The set W is countable so we may select an 1–1 coding function $\sigma_{\mathcal{F}} : W \to \Xi_2$ such that for every $(f_1, \dots, f_d) \in W$,

$$\sigma_{\mathcal{F}}(f_1, \dots, f_d) > \max\big\{k : \exists i \in \{1, \dots, d\} \text{ with } f_i \in F_k\big\}.$$

A finite or infinite block sequence $(f_i)_i$ in $\bigcup\limits_{j=1}^{\infty} F_j \setminus \{0\}$ is said to be a $\sigma_{\mathcal{F}}$ special sequence provided $f_1 \in \bigcup\limits_{l\in\Xi_1} F_l$ and $f_{i+1} \in F_{\sigma_{\mathcal{F}}(f_1,\dots,f_i)}$ for all i. A $\sigma_{\mathcal{F}}$ special functional x^* is any functional of the form $x^* = E\sum\limits_i f_i$ with $(f_i)_i$ a $\sigma_{\mathcal{F}}$ special sequence (when the sum $\sum\limits_i f_i$ is infinite it is considered in the pointwise topology) and E an interval of $\mathbb{N}$. If the interval E is finite then x^* is said to be a finite $\sigma_{\mathcal{F}}$ special functional. We denote by $\mathcal{S}$ the set of all finite $\sigma_{\mathcal{F}}$ special functionals. Let's observe that $\overline{\mathcal{S}}^{w^*}$ is the set of all $\sigma_{\mathcal{F}}$ special functionals.

Definition 3.3. (A) Let $s = (f_i)_i$ be a $\sigma_{\mathcal{F}}$ special sequence. Then for for each i we define the $\mathrm{ind}_s(f_i)$ as follows. $\mathrm{ind}_s(f_1) = \min\{j : f_1 \in F_j\}$ while for $i = 2, 3, \dots$ $\mathrm{ind}_s(f_i) = \sigma_{\mathcal{F}}(f_1, \dots, f_{i-1})$.

(B) Let $s = (f_i)_i$ be a $\sigma_{\mathcal{F}}$ special sequence and let E be an interval. The set of indices of the $\sigma_{\mathcal{F}}$ special functional $x^* = E\sum\limits_i f_i$ is the set $\mathrm{ind}_s(x^*) = \{\mathrm{ind}_s(f_i) : Ef_i \neq 0\}$.

(C) A (finite or infinite) family of $\sigma_{\mathcal{F}}$ special functionals $(x_k^*)_k$ is said to be disjoint if for each k there exists a $\sigma_{\mathcal{F}}$ special sequence $s_k = (f_i^k)_i$ and interval E_k such that $x_k^* = E_k\sum\limits_i f_i^k$ and $(\mathrm{ind}_{s_k}(x_k^*))_k$ are pairwise disjoint.

Remark 3.4. (a) Our definition of $\mathrm{ind}_s(f_i)$ and $\mathrm{ind}_s(x^*)$, which is rather technical, is required by the fact that we did not assume $(F_i \setminus \{0\})_i$ to be pairwise disjoint, hence the same f could occur in several different $\sigma_{\mathcal{F}}$ special sequences.

(b) Let's observe that for every family $(x_i^*)_{i=1}^d$ of disjoint $\sigma_{\mathcal{F}}$ special functionals, $\| \sum_{i=1}^{d} x_i^* \|_\infty \leq 1$ (recall that $\sum_{j=1}^{\infty} \tau_j \leq 1$).

(c) Let $s_1 = (f_i)_i$, $s_2 = (h_i)_i$ be two distinct $\sigma_{\mathcal{F}}$ special sequences. Then $\mathrm{ind}_{s_1}(f_i) \neq \mathrm{ind}_{s_2}(h_j)$ for $i \neq j$ while there exists i_0 such that $f_i = h_i$ for all $i < i_0$ and $\mathrm{ind}_{s_1}(f_i) \neq \mathrm{ind}_{s_2}(h_i)$ for $i > i_0$.

(d) For every family $(s_i)_{i=1}^d$ of infinite $\sigma_{\mathcal{F}}$ special sequences there exists n_0 such that $(Es_i^*)_{i=1}^d$ are disjoint, where $E = [n_0, \infty)$ and s_i^* denotes the $\sigma_{\mathcal{F}}$ special functional defined by the $\sigma_{\mathcal{F}}$ special sequence s_i.

Definition 3.5. (The norming set $\mathcal{F}_2$). Let $(F_j)_{j=0}^\infty$ be a JTG family. We set

$$\begin{aligned} \mathcal{F}_2 \quad = \quad & F_0 \cup \Big\{ \sum_{k=1}^{d} a_k x_k^* : \ a_k \in \mathbb{Q}, \ \sum_{k=1}^{d} a_k^2 \leq 1, \ \text{ and} \\ & (x_k^*)_{k=1}^d \text{ is a family of disjoint finite } \sigma_{\mathcal{F}} \text{ special functionals} \Big\} \end{aligned}$$

The space $JT_{\mathcal{F}_2}$ is defined as the completion of the space $(c_{00}(\mathbb{N}) \ , \| \ \|_{\mathcal{F}_2})$ where $\|x\|_{\mathcal{F}_2} = \sup\{f(x) : \ f \in \mathcal{F}_2\}$ for $x \in c_{00}(\mathbb{N})$.

Remark 3.6. Let's observe that the standard basis $(e_n)_{n\in\mathbb{N}}$ of $c_{00}(\mathbb{N})$ is a normalized bimonotone Schauder basis of the space $JT_{\mathcal{F}_2}$.

Theorem 3.7. (i) The space $JT_{\mathcal{F}_2}$ does not contain ℓ_1.
(ii) $JT_{\mathcal{F}_2}^* = \overline{\mathrm{span}}(\{e_n^* : \ n \in \mathbb{N}\} \cup \{b^* : \ b \in \mathcal{B}\})$ where $\mathcal{B}$ is the set of all infinite $\sigma_{\mathcal{F}}$ special sequences.

The proof of the above theorem is almost identical with the proofs of Propositions 10.4 and 10.11 of [AT1]. We proceed to a short description of the basic steps.

Let's start by observing the following.

$$\begin{aligned} \overline{\mathcal{F}_2}^{w^*} \quad = \quad & F_0 \cup \Big\{ \sum_{k=1}^{\infty} a_k x_k^* : \ a_k \in \mathbb{Q}, \ \sum_{k=1}^{\infty} a_k^2 \leq 1, \ \text{ and} \\ & (x_k^*)_{k=1}^\infty \text{ is a family of disjoint } \sigma_{\mathcal{F}} \text{ special functionals} \Big\} \end{aligned}$$

Also for a disjoint family $(x_i^*)_{i=1}^\infty$ of special functionals and $(a_i)_{i=1}^\infty$ in $\mathbb{R}$, we have that
$\| \sum_{i=1}^{\infty} a_i x_i^* \|_{JT_{\mathcal{F}_2}^*} \leq \big(\sum_{i=1}^{\infty} a_i^2 \big)^{1/2}$. The above observations yield the following:

Lemma 3.8. $\overline{\mathcal{F}}_2^{w^*} \subset \overline{\mathrm{span}}(\{e_n^* : \ n \in \mathbb{N}\} \cup \{b^* : \ b \in \mathcal{B}\})$ where $\mathcal{B}$ is the set of all infinite $\sigma_{\mathcal{F}}$ special sequences.

Observe also that $\overline{\mathcal{F}_2}^{w^*}$ is w^* compact and 1-norming hence contains the set $\mathrm{Ext}(B_{JT_{\mathcal{F}_2}^*})$. Rainwater's theorem and the above results yield that a bounded sequence $(x_k)_{k\in\mathbb{N}}$ is weakly Cauchy if and only if $\lim_k e_n^*(x_k)$ and $\lim_k b^*(x_k)$ exist for all n and infinite special sequences b. This is established by the following.

Lemma 3.9. Let $(x_k)_{k\in\mathbb{N}}$ be a bounded sequence in $JT_{\mathcal{F}_2}$ and let $\varepsilon > 0$. Then there exists a finite family $x_1^*, \dots, x_d^*$ of disjoint special functionals and an $L \in [\mathbb{N}]$ such that

$$\limsup_{k\in L} |x^*(x_k)| \leq \varepsilon$$

for every special functional x^* such that the family $x^*, x_1^*, \dots, x_d^*$ is disjoint.

For a proof we refer the reader to the proof of Lemma 10.5 [AT1].

Lemma 3.10. Let $(x_k)_{k\in\mathbb{N}}$ be a bounded sequence in $JT_{\mathcal{F}_2}$. There exists an $M \in [\mathbb{N}]$ such that for every special functional x^* the sequence $(x^*(x_k))_{k\in M}$ converges.

Also for the proof of this we refer the reader to the proof of Lemma 10.6 of [AT1]

Proof of Theorem 3.7. (i) Let $(x_k)_{k\in\mathbb{N}}$ be a bounded sequence in $JT_{\mathcal{F}_2}$. By an easy diagonal argument we may assume that for every $n \in \mathbb{N}$, $\lim_n e_n^*(x_k)$ exists. Lemma 3.10 also yields that there exists a subsequence $(x_{l_k})_{k\in\mathbb{N}}$ such that for every special sequence b, $\lim_k b^*(x_{l_k})$ also exists. As we have mentioned above Lemma 3.8 yields that $(x_{l_k})_{k\in\mathbb{N}}$ is weakly Cauchy.
(ii) Since $\mathrm{Ext}(B_{JT^*_{\mathcal{F}_2}}) \subset \overline{\mathcal{F}_2}^{w^*}$ and ℓ_1 does not embed into $JT_{\mathcal{F}_2}$ Haydon's theorem [Ha] yields that $\overline{\mathcal{F}_2}^{w^*}$ norm generates $JT^*_{\mathcal{F}_2}$. Lemma 3.8 yields the desired result. □

The remaining part of this section concerns the proof of Proposition 3.14, stated below. This will be used in the next section to show that a specific ground set $\mathcal{F}_2$ admits a strongly strictly singular extension.

Definition 3.11. Let $(x_n)_n$ be a bounded block sequence in $JT_{\mathcal{F}_2}$ and $\varepsilon > 0$. We say that $(x_n)_n$ is *ε-separated* if for every $\phi \in \cup_{j\in\mathbb{N}} F_j$

$$\#\{n : |\phi(x_n)| \geq \varepsilon\} \leq 1.$$

In addition, we say that $(x_n)_n$ is *separated* if for every $L \in [\mathbb{N}]$ and $\varepsilon > 0$ there exists an $M \in [L]$ such that $(x_n)_{n\in M}$ is ε-separated.

Lemma 3.12. Let $(x_n)_n$ be a bounded separated sequence in $JT_{\mathcal{F}_2}$ such that for every infinite $\sigma_{\mathcal{F}}$ special functional b^* we have that $\lim_n b^*(x_n) = 0$. Then for every $\varepsilon > 0$, there exists an $L \in [\mathbb{N}]$ such that for all $y^* \in \overline{\mathcal{S}}^{w^*}$,

$$\#\{n \in L : \ |y^*(x_n)| \geq \varepsilon\} \leq 2.$$

Proof. Assume the contrary and fix an $\varepsilon > 0$ such that the statement of the lemma is false. Define

$$A = \left\{(n_1 < n_2 < n_3) \in [\mathbb{N}]^3 : \ \exists y^* \in \overline{\mathcal{S}}^{w^*}, \ |y^*(x_{n_1})|, |y^*(x_{n_2})|, |y^*(x_{n_2})| \geq \varepsilon\right\}$$

and $B = [\mathbb{N}]^3 \setminus A$. Then Ramsey's Theorem yields that there exists an $L \in [\mathbb{N}]$ such that either $[L]^3 \subset A$ or $[L]^3 \subset B$. Our assumption rejects the second case, so we conclude that for all $n_1 < n_2 < n_3 \in L$, there is a $y^*_{n_1,n_2,n_3} \in \overline{\mathcal{S}}^{w^*}$ such that $|y^*_{n_1,n_2,n_3}(x_{n_i})| \geq \varepsilon$, $i = 1, 2, 3$.

Since $(x_n)_{n\in L}$ is separated, we may assume by passing to a subsequence that for $\varepsilon' = \frac{\varepsilon}{8}$, $(x_n)_{n\in L}$ is ε'-separated. For reasons of simplicity in the notation we may moreover and do assume that $(x_n)_{n\in\mathbb{N}}$ has both properties.

For all triples $(1 < n < k)$, let $y^*_{n,k}$ denote an element in $\overline{\mathcal{S}}^{w^*}$ such that $|y^*_{n,k}(x_i)| \geq \varepsilon$, $i = 1, n, k$. Moreover, let $y^*_{n,k} = E_{n,k}\sum_{i=1}^{\infty} \phi^i_{n,k}$ where $(\phi^i_{n,k})_{i\in\mathbb{N}}$ is a $\sigma_{\mathcal{F}}$ special sequence and $E_{n,k} \subset \mathbb{N}$ is an interval. For $1 < n < k$ we define the number $[n,k]$ as follows:

$$[n,k] = \min\{i \in \mathbb{N} : \ \max\operatorname{supp}\phi^i_{n,k} \geq \min\operatorname{supp} x_k\}.$$

Also, let $A = \{(n<k) \in [\mathbb{N}\setminus\{1\}]^2 : \ |\phi^{[n,k]}_{n,k}(x_n)| \leq \varepsilon'\}$ and $B = [\mathbb{N}\setminus\{1\}]^2 \setminus A$.

Again, using Ramsey's theorem and passing to a subsequence, we may and do assume that $[\mathbb{N}\setminus\{1\}]^2 \subset A$ or $[\mathbb{N}\setminus\{1\}]^2 \subset B$. Notice in the second case, that since $(x_n)_n$ is ε'-separated, we have that for all $1 < n < k$, $|\phi^{[n,k]}_{n,k}(x_k)| \leq \varepsilon'$. We set

$$s_{n,k} = \begin{cases} (\phi^1_{n,k}, \ldots, \phi^{[n,k]-1}_{n,k}), & \text{if } [\mathbb{N}\setminus\{1\}]^2 \subset A \\ (\phi^1_{n,k}, \ldots, \phi^{[n,k]}_{n,k}), & \text{if } [\mathbb{N}\setminus\{1\}]^2 \subset B \end{cases}.$$

Claim. There is an $M > 0$ such that for all $k \in \mathbb{N}$,

$$\#\{s_{n,k} : \ 2 \leq n \leq k-1\} \leq M.$$

Let $(x_n)_n$ be bounded by some $c > 0$. Next fix any $k \in \mathbb{N}$ and consider the following two cases:

The first case is $[\mathbb{N}\setminus\{1\}]^2 \subset B$. In this case $\phi^{[n,k]}_{n,k} \in s_{n,k}$ and if $s_{n_1,k} \neq s_{n_2,k}$ then $\phi^{[n_1,k]}_{n_1,k}$ is incomparable to $\phi^{[n_2,k]}_{n_2,k}$ in the sense,that every two special functionals, extending $s_{n_1,k}$ and $s_{n_2,k}$ respectively, have disjoint sets of indices.

So let $s_{n_j,k}$, $1 \leq j \leq N$ all be different from each other and consider the $\sigma_{\mathcal{F}}$ special functionals $z^*_{n_j} = E_{n_j,k} y^*_{n_j,k}$, $1 \leq j \leq N$ where $E_{n_j,k} = (\max\operatorname{supp}\phi^{[n_j,k]}_{n_j,k}, \infty)$. According to the previous observation these functionals have pairwise disjoint indices. Moreover

$$|z^*_{n_j}(x_k)| = |y^*_{n_j,k}(x_k) - \phi^{[n_j,k]}_{n_j,k}(x_k)| \geq \varepsilon - \varepsilon' \tag{3}$$

since $[\mathbb{N}\setminus\{1\}]^2 \subset B$ and $(x_n)_n$ is ε'-separated.

Inequality (3) yields that

$$\Big(\sum_{j=1}^{N} (z^*_{n_j}(x_k))^2\Big)^{1/2} \geq (\varepsilon - \varepsilon')N^{1/2}.$$

Therefore there are $(a_j)_{j=1}^N$ with $\sum_{j=1}^n a_j^2 \leq 1$ such that

$$\sum_{j=1}^{N} a_j z^*_{n_j}(x_k) \geq (\varepsilon - \varepsilon')N^{1/2}.$$

On the other hand, by the definition of the norm on $JT_{\mathcal{F}_2}$, $\sum_{j=1}^N a_j z^*_{n_j}(x_k) \leq \|x_k\| \leq c$. It follows that $N \leq (\frac{c}{\varepsilon - \varepsilon'})^2$ and this is the required upper estimate for N.

The second case is $[\mathbb{N}\backslash\{1\}]^2 \subset A$. As in the first case, if $1 < n_1 < n_2 < k$ and $s_{n_1,k} \neq s_{n_2,k}$ then $\phi_{n_1,k}^{[n_1,k]}$ and $\phi_{n_2,k}^{[n_2,k]}$ are incomparable and since $s_{n_1,k} \neq s_{n_2,k}$ they also have different indices. As in the first case let $s_{n_j,k}$, $1 \leq j \leq N$ all be different from each other and set $z^*_{n_j} E_{n_j,k} y^*_{n_j,k}$, $1 \leq j \leq N$ where in this case $E_{n,k} = [\min \operatorname{supp} \phi_{n_j,k}^{[n_j,k]}, \infty)$. By our previous observation it follows that these $\sigma_{\mathcal{F}}$ special functionals have pairwise disjoint indices. Notice also that $|z^*_{n_j}(x_k)| = |y^*_{n_j,k}(x_k)| \geq \varepsilon$. Therefore exactly as in the first case we obtain an upper estimate for N independent of k and this finishes the proof of the claim.

In the case where $[\mathbb{N} \setminus \{1\}]^2 \subset B$, $|s^*_{n,k}(x_n)| = |y^*_{n,k}(x_n)| \geq \varepsilon$.

In the case where $[\mathbb{N}\backslash\{1\}]^2 \subset A$, $|s^*_{n,k}(x_n)| = |y^*_{n,k}(x_n) - \phi_{n,k}^{[n,k]}(x_n)| \geq \varepsilon - \varepsilon'$.

In any case we have that $|s^*_{n,k}(x_n)| \geq \varepsilon - \varepsilon' > 0$.

Combining this with the previous claim, we get that for any $k \geq 3$ there are $z^*_{1,k}, \ldots, z^*_{M,k} \in \overline{\mathbb{S}}^{w^*}$ such that for any $1 < n < k$ there is $i \in [1, M]$ so that $|z^*_{i,k}(x_n)| \geq \varepsilon - \varepsilon'$.

Since $\overline{\mathbb{S}}^{w^*}$ is weak-* compact we can pass to an $L \in [\mathbb{N}]$ such that $(z^*_{i,k})_{k\in L}$ is weak-* convergent to some $z^*_i \in \overline{\mathbb{S}}^{w^*}$. It is easy to see that in this case, for any $n \in \mathbb{N}$ there is an $i \in [1, M]$ so that $|z^*_i(x_n)| \geq \varepsilon - \varepsilon'$. Therefore there exists an infinite subset P of $\mathbb{N}$ and $1 \leq i_0 \leq M$ such that $|z^*_{i_0}(x_n)| \geq \varepsilon - \varepsilon'$ for every $n \in P$. It also follows that $z^*_{i_0}$ is an infinite $\sigma_{\mathcal{F}}$ special functional. These contradict our assumption that $\lim_n b^*(x_n) = 0$ for every infinite $\sigma_{\mathcal{F}}$ special functional b^*. □

We now prove the following lemma about $JT_{\mathcal{F}_2}$:

Lemma 3.13. Let $x \in JT_{\mathcal{F}_2}$ with finite support and $\varepsilon > 0$. There exists $n \in \mathbb{N}$ such that if $y^* \sum_{k=1}^d a_k y^*_k \in \mathcal{F}_2$ with $\max\{|a_k| : 1 \leq k \leq d\} < \frac{1}{n}$, then $|y^*(x)| < \varepsilon$.

Proof. Let $\delta = \frac{\varepsilon}{2\sum_{n\in\mathbb{N}} |x(n)|}$ and $\tau_j = \sup\{\|f\|_\infty : f \in F_j\}$. Since $\sum_{j=1}^\infty \tau_j \leq 1$, by the definition of a JTG family, there is a $j_0 \in \mathbb{N}$ such that $\sum_{j=j_0+1}^\infty \tau_j < \delta$. Let n be such that $\frac{1}{n} < \frac{\varepsilon}{2j_0\|x\|}$.

Assume that $y^* = \sum_{k=1}^d a_k y^*_k \in \mathcal{F}_2$ with $\max\{|a_k| : 1 \leq k \leq d\} < \frac{1}{n}$. For every $k \in [1, d]$ let $y^*_k = y^*_{k,1} + y^*_{k,2}$ with $\operatorname{ind}(y^*_{k,1}) \subset \{1, \ldots, j_0\}$ and $\operatorname{ind}(y^*_{k,2}) \subset \{j_0 + 1, j_0 + 2, \ldots\}$. So we may write $y^* = \sum_{k=1}^d a_k y^*_{k,1} + \sum_{k=1}^d a_k y^*_{k,2}$. Notice now that for any $n \in \mathbb{N}$, $|\sum_{k=1}^d a_k y^*_{k,2}(n)| \leq \sum_{k=1}^d \|y^*_{k,2}\|_\infty$ and since $(\operatorname{ind}(y^*_{k,2}))_{k=1}^d$ are pairwise disjoint and all greater than j_0 we get that $\sum_{k=1}^d \|y^*_{k,2}\|_\infty < \delta$. Therefore $\|\sum_{k=1}^d a_k y^*_{k,2}\|_\infty < \delta$ and it follows that

$$\Big|\sum_{k=1}^d a_k y^*_{k,2}(x)\Big| \leq \sum_{n\in\mathbb{N}} \delta|x(n)| = \frac{\varepsilon}{2}. \tag{4}$$

On the other hand since $(y^*_{k,1})_{k=1}^d$ have pairwise disjoint indices, at most j_0 of them are non-zero and $|y^*_{k,1}(x)| \leq \|x\|$. Therefore $|\sum_{k=1}^d a_k y^*_{k,1}(x)| \leq j_0 \frac{1}{n}\|x\| < \frac{\varepsilon}{2}$. Combining this with (4) we get that $|y^*(x)| < \varepsilon$ as required. □

We combine now Lemmas 3.12 and 3.13 to prove the following:

Proposition 3.14. Let $(x_n)_n$ be a weakly null separated sequence in $JT_{\mathcal{F}_2}$ with $\|x_n\|_{\mathcal{F}_2} \leq C$ for all n. Then for all $m \in \mathbb{N}$, there is $L \in [\mathbb{N}]$ such that for every $y^* \in \mathcal{F}_2$,

$$\#\{n \in L : |y^*(x_n)| \geq \frac{1}{m}\} \leq 66m^2C^2.$$

Proof. We may and do assume that $\{x_n : n \in \mathbb{N}\}$ is normalized. We set $\delta_1 = \frac{1}{4m}$ and we find $L_1 \in [\mathbb{N}]$ such that Lemma 3.12 is valid for $\varepsilon = \delta_1$. Then we set $n_1 = \min L_1$ and using Lemma 3.13 we find $n = r_1 \in \mathbb{N}$ such that the conclusion of Lemma 3.13 is valid for $\varepsilon = \delta_1$ and $x = x_{n_1}$. Then, after setting $\delta_2 = \min\{\frac{1}{8mr_1^2}, \delta_1\}$ we find $L_2 \in [\mathbb{N} \setminus \{n_1\}]$ such that Lemma 3.12 is valid for $\varepsilon = \delta_2$. We set $n_2 = \min L_2$ and we find $n = r_2 \in \mathbb{N}$ with $r_2 > r_1$ such that the conclusion of Lemma 3.13 is valid for $\varepsilon = \delta_2$ and $x \in \{x_{n_1}, x_{n_2}\}$.

Recursively, having defined $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_{p-1}$, $L_1 \supset L_2 \supset L_3 \supset \cdots \supset L_{p-1}$, $n_1 < n_2 < \cdots < n_{p-1}$ and $r_1 < r_2 < \cdots < r_{p-1}$, we set $\delta_p = \min\{\frac{1}{4m2^{p-1}r_{p-1}^2}, \delta_{p-1}\}$ and we find $L_p \in [L_{p-1} \setminus \{n_{p-1}\}]$ such that Lemma 3.12 is valid for $\varepsilon = \delta_1$. We set $n_p = \min L_p$ and we find $n = r_p > r_{p-1}$ that the conclusion of Lemma 3.13 is satisfied for $\varepsilon = \delta_p$ and $x \in \{x_{n_1}, x_{n_2}, \ldots, x_{n_p}\}$. At the end we consider the set $L = \{n_1 < n_2 < \cdots < n_p < \cdots\}$.

The crucial properties of this construction are the following:

(1) If $\sum_{k=1}^\ell a_k y_k^* \in \mathcal{F}_2$ and $|a_k| < \frac{1}{r_p}$, for $k = 1, \ldots, \ell$ then we have that $|\sum_{k=1}^\ell a_k y_k^*(x_{n_i})| < \delta_p$ for all $i = 1, \ldots, p$.

(2) For every $x^* \in \mathcal{S}$ we have that $\#\{i \geq p : |x^*(x_{n_i})| \geq \delta_p\} \leq 2$.

We will make use of these two properties to prove the proposition.

Let $d = 66m^2$. It suffices to prove that if $n_{\ell_1} < n_{\ell_2} < \cdots < n_{\ell_d}$ and $y^* = \sum_{k=1}^\ell a_k y_k^* \in \mathcal{F}_2$, then there is an $1 \leq i \leq d$ such that $|y^*(x_{n_{\ell_i}})| < \frac{1}{m}$.

We set

$$A_1 = \{k \in [1, \ell] : |a_k| \geq \frac{1}{r_{\ell_1}}\},$$

$$A_p = \{k \in [1, \ell] : \frac{1}{r_{\ell_p}} \leq |a_k| < \frac{1}{r_{\ell_{p-1}}}\}, \quad \text{for } 1 < p < d$$

$$A_d = \{k \in [1, \ell] : \frac{1}{r_{\ell_d}} > |a_k|\}.$$

Observe that for $p < d$, we have that $\#A_p \leq r_{\ell_p}^2$. By property (1) we have that for any $p \in [1, d)$,

$$\Big|\sum_{k \in \cup_{j>p} A_j} a_k y_k^*(x_{n_{\ell_p}})\Big| < \delta_{\ell_p} \leq \delta_1 \frac{1}{4m}. \tag{5}$$

Next, for $1 \leq j < p \leq d$ we set

$$B_{j,p} = \{k \in A_j : |y_k^*(x_{n_{\ell_p}})| \geq \delta_{\ell_j + 1}\}$$

and then for $p \in (1,d]$ we define

$$B_p = \bigcup_{j<p} B_{j,p}.$$

Since for $p > j$ we have that $\ell_p \geq \ell_j + 1$, property (2) yields that for every $k \in A_j$ there exist at most two $B_{j,p}$'s containing k. Hence every $k \in \{1, \ldots, \ell\}$ belongs to at most two B_p's.

Next we shall estimate the term $\sum_{k \in \cup_{j<p} A_j \setminus B_p} |a_k y_k^*(x_{n_{\ell_p}})|$. Let $p \in (1,d]$. We have that

$$\begin{aligned} \sum_{k \in \cup_{j<p} A_j \setminus B_p} |a_k y_k^*(x_{n_{\ell_p}})| &= \sum_{j=1}^{p-1} \sum_{k \in A_j \setminus B_{j,p}} |a_k y_k^*(x_{n_{\ell_p}})| \\ &\leq \sum_{j=1}^{p-1} \#(A_j) \delta_{\ell_j+1} \leq \sum_{j=1}^{p-1} r_{\ell_j}^2 \frac{1}{4m 2^{\ell_j} r_{\ell_j}^2} < \frac{1}{4m}. \end{aligned} \tag{6}$$

We now argue that for at least $\frac{d}{2}+1$ many of $\{x_{n_{\ell_1}}, \ldots, x_{n_{\ell_d}}\}$, we have that $|\sum_{k \in A_p} a_k y_k^*(x_{n_{\ell_p}})| < \frac{1}{4m}$. If this is not the case, then for at least $\frac{d}{2}$ many, $|\sum_{k \in A_p} a_k y_k^*(x_{n_{\ell_p}})| \geq \frac{1}{4m}$ and therefore $\sum_{k \in A_p} a_k^2 \geq \frac{1}{16m^2}$. Thus

$$\sum_{k=1}^{\ell} a_k^2 \sum_{p=1}^{d} \sum_{k \in A_p} a_k^2 \geq \frac{d}{2} \cdot \frac{1}{16m^2} = \frac{d}{32m^2}$$

which is a contradiction since $d = 66m^2$ and $\sum_{k=1}^{\ell} a_k^2 \leq 1$.

Now we shall prove that for at least $\frac{d}{2}+1$ many of $\{x_{n_{\ell_1}}, \ldots, x_{n_{\ell_d}}\}$, $|\sum_{k \in B_p} a_k y_k^*(x_{n_{\ell_p}})| < \frac{1}{4m}$. Again if this is not the case, then for at least $\frac{d}{2}$ many $|\sum_{k \in B_p} a_k y_k^*(x_{n_{\ell_p}})| \geq \frac{1}{4m}$ and therefore $\sum_{k \in B_p} a_k^2 \geq \frac{1}{16m^2}$. Since every k appears in at most two B_p's, we have that

$$2 \geq 2 \sum_{k=1}^{\ell} a_k^2 \geq \sum_{p=1}^{d} \sum_{k \in B_p} a_k^2 \geq \frac{d}{2} \cdot \frac{1}{16m^2} = \frac{d}{32m^2}$$

which is a contradiction.

These last two observations show that there exists at least one $p \in [1,d]$ such that both $|\sum_{k \in A_p} a_k y_k^*(x_{n_{\ell_p}})| < \frac{1}{4m}$ and $|\sum_{k \in B_p} a_k y_k^*(x_{n_{\ell_p}})| < \frac{1}{4m}$.

Combining this with (5) and (6), we get that for this particular p,

$$\begin{aligned} \Big| \sum_{k=1}^{\ell} a_k y_k^*(x_{n_{\ell_p}}) \Big| \leq & \Big| \sum_{k \in \cup_{j>p} A_j} a_k y_k^*(x_{n_{\ell_p}}) \Big| + \Big| \sum_{k \in A_p} a_k y_k^*(x_{n_{\ell_p}}) \Big| \\ & + \Big| \sum_{k \in B_p} a_k y_k^*(x_{n_{\ell_p}}) \Big| + \Big| \sum_{k \in \cup_{j<p} A_j \setminus B_p} a_k y_k^*(x_{n_{\ell_p}}) \Big| \\ < & \frac{1}{m} \end{aligned}$$

as required. □

4. The space $(\mathfrak{X}_{\mathcal{F}_2})_*$ and the space of the operators $\mathcal{L}((\mathfrak{X}_{\mathcal{F}_2})_*)$

In this section we proceed to construct a HI space not containing a reflexive subspace. This space is $(\mathfrak{X}_{\mathcal{F}_2})_*$ where $\mathfrak{X}_{\mathcal{F}_2}$ is the strongly strictly singular HI extension (Sections 1 and 2) of the set $\mathcal{F}_2$. The set $\mathcal{F}_2$ is defined from a family $F = (F_j)_j$ as in Section 3. The proof that $(\mathfrak{X}_{\mathcal{F}_2})_*$ does not contain a reflexive subspace, uses the method of attractors and the key ingredient is the attractor functional and the attracting sequences introduced in Section 1. The structure of the quotients of $\mathfrak{X}_{\mathcal{F}_2}$ is also investigated.

The family $F = (F_j)_{j\in\mathbb{N}}$

We shall use the sequence of positive integers $(m_j)_j$, $(n_j)_j$ introduced in Definition 1.2 of strictly singular extensions which for convenience we recall:

- $m_1 = 2$ and $m_{j+1} = m_j^5$.
- $n_1 = 4$, and $n_{j+1} = (5n_j)^{s_j}$ where $s_j = \log_2 m_{j+1}^3$.

We set $F_0 = \{\pm e_n^* :\ n \in \mathbb{N}\}$ and for $j = 1, 2, \ldots$ we set

$$F_j = \Big\{\frac{1}{m_{4j-3}^2}\sum_{i\in I} \pm e_i^* :\ \#(I) \le \frac{n_{4j-3}}{2}\Big\} \cup \{0\}.$$

In the sequel we shall denote by $\mathfrak{X}_{\mathcal{F}_2}$ the HI extension of $JT_{\mathcal{F}_2}$ with ground set $\mathcal{F}_2$ defined by the aforementioned family $(F_j)_{j\in\mathbb{N}}$ as in Definition 3.5.

Proposition 4.1. The space $\mathfrak{X}_{\mathcal{F}_2}$ is a strongly strictly singular extension of $JT_{\mathcal{F}_2}(= Y_{\mathcal{F}_2})$.

Proof. Let $C > 0$. We select $j(C)$ such that $\frac{33}{2}m_{2j}^4C^2 < n_{2j-1}$ for every $j \ge j(C)$ and we shall show that the integer $j(C)$ satisfies the conclusion of Definition 2.1.

Let $(x_n)_{n\in\mathbb{N}}$ be a block sequence in $\mathfrak{X}_{\mathcal{F}_2}$ such that $\|x_n\| \le C$ for all n, $\|x_n\|_\infty \to 0$ and $(x_n)_{n\in\mathbb{N}}$ is a weakly null sequence in $JT_{\mathcal{F}_2}$. It suffices to show that the sequence $(x_n)_{n\in\mathbb{N}}$ is separated (Definition 3.11). Indeed, then Proposition 3.14 and our choice of $j(C)$ yield that for every $j \ge j(C)$ there exists $L \in [\mathbb{N}]$ such that for every $y^* \in \mathcal{F}_2$ we have that $\#\{n \in L :\ |y^*(x_n)| > \frac{2}{m_{2j}^2}\} \le 66(\frac{m_{2j}^2}{2})^2C^2 < n_{2j-1}$.

In order to show that the sequence $(x_n)_{n\in\mathbb{N}}$ is separated we start with the following easy observations:

(i) If $m_{4j_0-3}^2 > \frac{C}{\varepsilon}\#\operatorname{supp}(x)$ and $\|x\| \le C$ then for every $\phi \in \bigcup_{j\ge j_0} F_j$ we have that $|\phi(x)| \le \varepsilon$.

(ii) If $\|x\|_\infty < \frac{2\varepsilon}{n_{4j_0-3}}$ and $\phi \in \bigcup_{j\le j_0} F_j$ then $|\phi(x)| \le \varepsilon$.

Let $L \in [\mathbb{N}]$ and $\varepsilon > 0$. Using (i) and (ii) we may inductively select $1 = j_0 < j_1 < j_2 < \cdots$ in $\mathbb{N}$ and $k_1 < k_2 < \cdots$ in L such that for each i and $\phi \in \bigcup_{j\notin[j_{i-1},j_i)} F_j$ we have that $|\phi(x_{k_i})| < \varepsilon$. Setting $M = \{k_1, k_2, \ldots\}$ we have that the sequence $(x_n)_{n\in M}$ is ε-separated. Therefore the sequence $(x_n)_{n\in\mathbb{N}}$ is separated. $\square$

A consequence of the above proposition and the results of Sections 1 and 2 is the following:

Theorem 4.2. (a) The space $\mathfrak{X}_{\mathcal{F}_2}$ is HI and reflexively saturated.
(b) The predual $(\mathfrak{X}_{\mathcal{F}_2})_*$ is HI.
(c) Every bounded linear operator $T : \mathfrak{X}_{\mathcal{F}_2} \to \mathfrak{X}_{\mathcal{F}_2}$ is of the form $T = \lambda I + S$, with S strictly singular and weakly compact.
(d) Every bounded linear operator $T : (\mathfrak{X}_{\mathcal{F}_2})_* \to (\mathfrak{X}_{\mathcal{F}_2})_*$ is of the form $T = \lambda I + S$, with S strictly singular.

Proof. All the above properties are consequences of the fact that $\mathfrak{X}_{\mathcal{F}_2}$ is a strongly strictly singular extension of $JT_{\mathcal{F}_2}$. In particular (a) follows from Proposition 1.21 and Theorem 1.18, (b) follows from Theorem 2.12, (c) follows from Theorem 2.15 while (d) follows from Theorem 2.16. $\square$

Proposition 4.3. Let $(x_k, x_k^*)_{k=1}^{n_{4j-3}}$ be a $(18, 4j-3, 1)$ attracting sequence in $\mathfrak{X}_{\mathcal{F}_2}$ such that $\|x_{2k-1}\|_{\mathcal{F}_2} \le \frac{2}{m_{4j-3}^2}$ for $k = 1, \ldots, n_{4j-3}/2$. Then

$$\|\frac{1}{n_{4j-3}} \sum_{k=1}^{n_{4j-3}} (-1)^{k+1} x_k\| \le \frac{144}{m_{4j-3}^2}.$$

Proof. The conclusion follows by an application of Proposition 1.17 (ii) after checking that for every $g \in \mathcal{F}_2$ we have that $|g(x_{2k})| > \frac{2}{m_{4j-3}^2}$ for at most n_{4j-4} k's. From the fact that each x_{2k} is of the form e_l it suffices to show that for every $g \in \mathcal{F}_2$, the cardinality of the set $\{l : \ |g(e_l)| > \frac{2}{m_{4j-3}^2}\}$ is at most n_{4j-4}.

Let $g \in \mathcal{F}_2$, $g = \sum_{i=1}^{d} a_i g_i$ where $\sum_{i=1}^{d} a_i^2 \le 1$ and $(g_i)_{i=1}^d$ are $\sigma_{\mathcal{F}}$ special functionals with disjoint indices. For each i we divide the functional g_i into two parts, $g_i = y_i^* + z_i^*$, with $\mathrm{ind}(y_i^*) \subset \{1, \ldots, j-1\}$ and $\mathrm{ind}(z_i^*) \subset \{j, j+1, \ldots\}$. For $l \notin \bigcup_{i=1}^{d} \mathrm{supp}(y_i^*)$ we have that $|g(e_l)| \le \sum_{i=1}^{d} |z_i^*(e_l)| < \sum_{r=j}^{\infty} \frac{1}{m_{4r-3}^2} < \frac{2}{m_{4j-3}^2}$. Since $\#\big(\bigcup_{i=1}^{d} \mathrm{supp}(y_i^*)\big) \le \frac{n_1}{2} + \frac{n_5}{2} + \cdots \frac{n_{4j-7}}{2} < n_{4j-5}$ the conclusion follows.

The proof of the proposition is complete. $\square$

Definition 4.4. Let $\chi = (x_k, x_k^*)_{k=1}^{n_{4j-3}}$ be a $(18, 4j-3, 1)$ attracting sequence, with $\|x_{2k-1}\|_{\mathcal{F}_2} \le \frac{2}{m_{4j-3}^2}$ for $1 \le k \le n_{4j-3}/2$. We set

$$g_\chi = \frac{1}{m_{4j-3}^2} \sum_{k=1}^{n_{4j-3}/2} x_{2k}^*$$

$$F_\chi = -\frac{1}{m_{4j-3}^2} \sum_{k=1}^{n_{4j-3}/2} x_{2k-1}^*$$

$$d_\chi = \frac{m_{4j-3}^2}{n_{4j-3}} \sum_{k=1}^{n_{4j-3}} (-1)^k x_k$$

Lemma 4.5. If χ is a $(18, 4j-3, 1)$ attracting sequence, $\chi = (x_k, x_k^*)_{k=1}^{n_{4j-3}}$, with $\|x_{2k-1}\|_{\mathcal{F}_2} \le \frac{2}{m_{4j-3}^2}$ for $1 \le k \le n_{4j-3}/2$ then

(1) $\|g_\chi - F_\chi\| \leq \frac{1}{m_{4j-3}}$.
(2) $\frac{1}{2} = g_\chi(d_\chi) \leq \|d_\chi\| \leq 144$, and hence $\|g_\chi\| \geq \frac{1}{288}$.

Proof. (1) We have $g_\chi - F_\chi = \frac{1}{m_{4j-3}}\big(\frac{1}{m_{4j-3}}\sum_{k=1}^{n_{4j-3}} x_k^*\big)$. Since $(x_k^*)_{k=1}^{n_{4j-3}}$ is a special sequence of length n_{4j-3}, the functional $\frac{1}{m_{4j-3}}\sum_{k=1}^{n_{4j-3}} x_k^*$ belongs to D_G and hence to $B_{\mathfrak{X}^*_{\mathcal{F}_2}}$ The conclusion follows.

(2) It is straightforward from Definitions 1.15 and 4.4 that $g_\chi(d_\chi) = \frac{1}{2}$. Since $\|x_{2k-1}\|_{\mathcal{F}_2} \leq \frac{2}{m^2_{4j-3}}$ for $k = 1, \ldots, n_{4j-3}/2$, Proposition 4.3 yields that $\|d_\chi\| \leq 144$. Thus $\|g_\chi\| \geq \frac{g_\chi(d_\chi)}{\|d_\chi\|} \geq \frac{\frac{1}{2}}{144} = \frac{1}{288}$. □

Lemma 4.6. Let Z be a block subspace of $(\mathfrak{X}_{\mathcal{F}_2})_*$. Also, let $\varepsilon > 0$ and $j > 1$. There exists a $(18, 4j-3, 1)$ attracting sequence $\chi = (x_k, x_k^*)_{k=1}^{n_{4j-3}}$ with $\sum_{k=1}^{n_{4j-3}/2} \|x_{2k-1}\|_{\mathcal{F}_2} < \frac{1}{n^2_{4j-3}}$ and $\mathrm{dist}(F_\chi, Z) < \varepsilon$.

Proof. We select an integer j_1 such that $m_{2j_1}^{\frac{1}{2}} > n_{4j-3}$. From Lemma 2.10 we may select a $(18, 2j_1, 1)$ exact pair (x_1, x_1^*) with $\mathrm{dist}(x_1^*, Z) < \frac{\varepsilon}{n_{4j-3}}$ and $\|x_1\|_{\mathcal{F}_2} \leq \frac{2}{m_{2j_1}}$. Let $2j_2 = \sigma(x_1^*)$. We select $l_2 \in \Lambda_{2j_2}$ and we set $x_2 = e_{l_2}$ and $x_2^* = e_{l_2}^*$.

We then set $2j_3 = \sigma(x_1^*, x_2^*)$ and we select, using Lemma 2.10, a $(18, 2j_3, 1)$ exact pair (x_3, x_3^*) with $x_2 < x_3$, $\mathrm{dist}(x_3^*, Z) < \frac{\varepsilon}{n_{4j-3}}$ and $\|x_3\|_{\mathcal{F}_2} \leq \frac{2}{m_{2j_3}}$.

It is clear that we may inductively construct a $(18, 4j-3, 1)$ attracting sequence $\chi = (x_k, x_k^*)_{k=1}^{n_{4j-3}}$ such that $\sum_{k=1}^{n_{4j-3}/2} \|x_{2k-1}\|_{\mathcal{F}_2} \leq \sum_{k=1}^{n_{4j-3}/2} \frac{2}{m_{2j_{2k-1}}} < \frac{1}{n^2_{4j-3}}$ and $\mathrm{dist}(x^*_{2k-1}, Z) < \frac{\varepsilon}{n_{4j-3}}$ for $1 \leq k \leq n_{4j-3}/2$. It follows that $\mathrm{dist}(F_\chi, Z) \leq \frac{1}{m^2_{4j-3}} \sum_{k=1}^{n_{4j-3}/2} \mathrm{dist}(x^*_{2k-1}, Z) < \varepsilon$. □

Theorem 4.7. The space $(\mathfrak{X}_{\mathcal{F}_2})_*$ is a Hereditarily James Tree (HJT) space. In particular it does not contain any reflexive subspace and every infinite dimensional subspace Z of $(\mathfrak{X}_{\mathcal{F}_2})_*$ has nonseparable second dual Z^{**}.

Proof. Since each subspace of $(\mathfrak{X}_{\mathcal{F}_2})_*$ has a further subspace isomorphic to a block subspace it is enough to consider a block subspace Z of $(\mathfrak{X}_{\mathcal{F}_2})_*$ and to show that Z has the James Tree property.

We select a $j_\emptyset \in \Xi_1$ with $j_\emptyset \geq 2$. We shall inductively construct a family $(\chi_a)_{a\in\mathcal{D}}$ of attracting sequences and a family $(j_a)_{a\in\mathcal{D}}$ of integers such that

(i) If $a <_{lex} \beta$ then $d_{\chi_a} < d_{\chi_\beta}$.
(ii) For every $\beta \in \mathcal{D}$, $\chi_\beta = \big(x_k^\beta, (x_k^\beta)^*\big)_{k=1}^{n_{4j_\beta-3}}$ is a $(18, 4j_\beta - 3, 1)$ attracting sequence with $\mathrm{dist}(F_{\chi_\beta}, Z) < \frac{1}{m_{4j_\beta-3}}$ and $\|x^\beta_{2k-1}\|_{\mathcal{F}_2} \leq \frac{2}{m^2_{4j_\beta-3}}$ for $k = 1, \ldots, n_{4j_\beta-3}/2$.
(iii) If $\beta \in \mathcal{D}$ with $\beta \neq \emptyset$ then $j_\beta = \sigma_{\mathcal{F}}((g_{\chi_a})_{a<\beta})$.

The induction runs on the lexicographical ordering of $\mathcal{D}$. In the first step, i.e. for $\beta = \emptyset$, we select a $(18, 2j_\emptyset - 1, 1)$ attracting sequence with $\text{dist}(F_{\chi_\emptyset}, Z) < \frac{1}{m_{4j_\emptyset - 3}}$ and $\|x^\emptyset_{2k-1}\|_{\mathcal{F}_2} \leq \frac{2}{m^2_{4j_\emptyset-3}}$ for $k = 1, \ldots, n_{4j_\emptyset - 3}/2$. In the general inductive step, we assume that $(j_a)_{a <_{lex} \beta}$ and $(\chi_a)_{a<_{lex}\beta}$ have been constructed for some $\beta \in \mathcal{D}$. Since $\{a \in \mathcal{D} : \ a < \beta\} \subset \{a \in \mathcal{D} : \ a <_{lex} \beta\}$, the attracting sequences $(\chi_a)_{a<\beta}$ have already been constructed so we may set $j_\beta = \sigma_{\mathcal{F}}((g_{\chi_a})_{a<\beta})$. Denoting by β^- the immediate predecessor of β in the lexicographical ordering, we select, using Lemma 4.6, a $(18, 2j_\beta - 1, 1)$ attracting sequence $\chi_\beta = \big(x^\beta_k, (x^\beta_k)^*\big)_{k=1}^{n_{4j_\beta-3}}$ with $d_{\chi_{\beta^-}} < d_{\chi_\beta}$ such that $\text{dist}(F_{\chi_\beta}, Z) < \frac{1}{m_{4j_\beta-3}}$ and $\|x^\beta_{2k-1}\|_{\mathcal{F}_2} \leq \frac{2}{m^2_{4j_\beta-3}}$ for $1 \leq k \leq n_{4j_\beta-3}/2$. The inductive construction is complete.

For each branch b of the dyadic tree the sequence $(g_{\chi_a})_{a\in b}$ is a $\sigma_{\mathcal{F}}$ special sequence. Thus the series $\sum_{a\in\mathcal{D}} g_{\chi_a}$ converges in the w^* topology to a $\sigma_{\mathcal{F}}$ special functional $g_b \in \overline{G}^{w^*} \subset \overline{D_G}^{w^*} = B_{\mathfrak{X}^*_{\mathcal{F}_2}}$.

For each $\beta \in \mathcal{D}$ we select a $z^*_\beta \in Z$ such that $\|z^*_\beta - F_{\chi_\beta}\| < \frac{1}{m_{4j_\beta-3}}$. Then Lemma 4.5 (1) yields that $\|z^*_\beta - g_{\chi_\beta}\| \leq \|z^*_\beta - F_{\chi_\beta}\| + \|F_{\chi_\beta} - g_{\chi_\beta}\| < \frac{2}{m_{4j_\beta-3}}$. Now let b be a branch of the dyadic tree. Since $\sum_{a\in b}\|z^*_a - g_{\chi_a}\| < \sum_{a\in b}\frac{2}{m_{4j_a-3}} < \frac{3}{m_{4j_\emptyset-3}} < \frac{1}{1152}$ it follows that the series $\sum_{a\in\mathcal{D}} z^*_a$ is also w^* convergent and its w^* limit $z^*_b \in Z^{**}$ satisfies $\|z^*_b - g_b\| < \frac{1}{1152}$. This actually yields that the block sequence $(z^*_a)_{a\in\mathcal{D}}$ defines a James Tree structure in the subspace Z.

The family $\{z^*_b : \ b \text{ a branch of } \mathcal{D}\}$ is a family in Z^{**} with the cardinality of the continuum. We complete the proof of the theorem by showing that for $b \neq b'$ we have that $\|z^*_b - z^*_{b'}\| \geq \frac{1}{576}$. Let $b \neq b'$ be two branches of the dyadic tree. We select $a \in \mathcal{D}$ with $a \in b \setminus b'$ (i.e. a is an initial part of b but not of b'). Then our construction and Lemma 4.5 (2) yield that

$$\begin{aligned}
\|z^*_b - z^*_{b'}\| &\geq \|g_b - g_{b'}\| - \|z^*_b - g_b\| - \|z^*_{b'} - g_{b'}\| \\
&> \frac{(g_b - g_{b'})(d_{\chi_a})}{\|d_{\chi_a}\|} - \frac{1}{1152} - \frac{1}{1152} \\
&\geq \frac{g_{\chi_a}(d_{\chi_a})}{144} - \frac{1}{576} = \frac{\frac{1}{2}}{144} - \frac{1}{576} = \frac{1}{576}.
\end{aligned}$$

□

Proposition 4.8. For every block subspace $Y = \overline{\text{span}}\{y_n : \ n \in \mathbb{N}\}$ of $\mathfrak{X}_{\mathcal{F}_2}$ there exist a further block subspace $Y' = \overline{\text{span}}\{y'_n : \ n \in \mathbb{N}\}$ and a block subspace $Z = \overline{\text{span}}\{z_k : \ k \in \mathbb{N}\}$ of $\mathfrak{X}_{\mathcal{F}_2}$ such that the following are satisfied. The space Z is reflexive, the spaces Y' and Z are disjointly supported (i.e. $\text{supp}\, z_k \cap \text{supp}\, y'_n = \emptyset$ for all n, k) and the space $X = \overline{\text{span}}(\{z_k : \ k \in \mathbb{N}\} \cup \{y'_n : n \in \mathbb{N}\})$ has nonseparable dual.

Proof. The proof is similar to that of Theorem 4.7. Let Y be a block subspace of $\mathfrak{X}_{\mathcal{F}_2}$. Using Proposition 1.14 we may inductively construct (the induction runs on the lexicographic order of the dyadic tree $\mathcal{D}$) a family $(\chi_a)_{a\in\mathcal{D}}$ of attracting sequences and a family $(j_a)_{a\in\mathcal{D}}$ of integers such that the following conditions are satisfied:

(i) If $a <_{lex} \beta$ then $d_{\chi_a} < d_{\chi_\beta}$.

(ii) For every $\beta \in \mathcal{D}$, $\chi_\beta = \big(x_k^\beta, (x_k^\beta)^*\big)_{k=1}^{n_{4j_\beta-3}}$ is a $(18, 2j_\beta - 1, 1)$ attracting sequence with $x_{2k-1}^\beta \in Y$ and $\|x_{2k-1}^\beta\|_{\mathcal{F}_2} \leq \frac{2}{m_{4j_\beta-3}^2}$ for $k = 1, \ldots, n_{4j_\beta-3}/2$.

(iii) $j_\emptyset \in \Xi_1$ with $j_\emptyset \geq 2$, while if $\beta \in \mathcal{D}$ with $\beta \neq \emptyset$, then $j_\beta = \sigma_{\mathcal{F}}((g_{\chi_\emptyset})_{a<\beta})$.

For each $a \in \mathcal{D}$ we set $z_a = \frac{2m_{4j_a-3}}{n_{4j_a-3}} \sum_{k=1}^{n_{4j_a-3}/2} x_{2k}^a$ and we consider the space $Z = \overline{\operatorname{span}}\{z_a : a \in \mathcal{D}\}$.

We first observe that for each $a \in \mathcal{D}$ the functional $f_a = \frac{1}{m_{4j_a-3}} \sum_{k=1}^{n_{4j_a-3}} (x_k^a)^*$ belongs to $D_G \subset B_{\mathfrak{X}_{\mathcal{F}_2}}$, hence $\|z_a\| \geq f_a(z_a) = 1$. On the other hand we have that $\|z_a\|_{\mathcal{F}_2} \leq \frac{2}{m_{4j_a-3}}$. Indeed, let $g = \sum_{i=1}^d a_i g_i \in \mathcal{F}_2$ (i.e. $\sum_{i=1}^d a_i^2 \leq 1$ while $(g_i)_{i=1}^d$ and $\sigma_{\mathcal{F}}$ special functionals with pairwise disjoint indices). For each $i = 1, \ldots, d$ let $g_i = y_i^* + z_i^*$ with $\operatorname{ind}(y_i^*) \subset \{1, \ldots, j_a - 1\}$ and $\operatorname{ind}(z_i^*) \subset \{j_a, j_a + 1, \ldots\}$. Then

$$
\begin{aligned}
|g(z_a)| &\leq \sum_{i=1}^d |y_i^*(z_a)| + \sum_{i=1}^d |z_i^*(z_a)| \\
&\leq \frac{2m_{4j_a-3}}{n_{4j_a-3}}\Big(\frac{n_1}{2} + \cdots + \frac{n_{4j_a-7}}{2}\Big) + \frac{2m_{4j_a-3}}{n_{4j_a-3}} \sum_{r=j_a}^{\infty} \frac{n_{4j_a-3}}{2} \frac{1}{m_{4r-3}^2} \\
&\leq \frac{2}{m_{4j_a-3}}.
\end{aligned}
$$

It follows that $\sum_{a\in\mathcal{D}} \frac{\|z_a\|_{\mathcal{F}_2}}{\|z_a\|} \leq \sum_{a\in\mathcal{D}} \frac{2}{m_{4j_a-3}} < \frac{1}{2}$ which yields that the space $Z = \overline{\operatorname{span}}\{z_a : a \in \mathcal{D}\}$ is reflexive (see Proposition 1.21).

For every branch b of the dyadic tree, the functional g_b which is defined to be the w^* sum of the series $\sum_{a\in\mathcal{D}} g_{\chi_a}$ belongs to $\overline{\mathcal{F}_2}^{w^*} \subset B_{\mathfrak{X}_{\mathcal{F}_2}}$. The family $\{g_b|_X : b \text{ a branch of } \mathcal{D}\}$ is a family of X^* with the cardinality of the continuum. For $b \neq b'$, selecting $a \in b \setminus b'$ the vector $d_{\chi_a} = \frac{m_{4j_a-3}^2}{n_{4j_a-3}} \sum_{k=1}^{n_{4j_a-3}} (-1)^k x_k^a$ belongs to $Y + Z$ while from Lemma 4.5 we have that $\|d_{\chi_a}\| \leq 144$. Thus $\|g_b|_X - g_{b'}|_X\|_{X^*} \geq \frac{g_b(d_{\chi_a}) - g_{b'}(d_{\chi_a})}{\|d_{\chi_a}\|} \geq \frac{\frac{1}{2}-0}{144} = \frac{1}{288}$.

Therefore X^* is nonseparable. □

Lemma 4.9. If $S : (\mathfrak{X}_{\mathcal{F}_2})_* \to (\mathfrak{X}_{\mathcal{F}_2})_*$ is a strictly singular operator then its conjugate operator $S^* : \mathfrak{X}_{\mathcal{F}_2} \to \mathfrak{X}_{\mathcal{F}_2}$ is also strictly singular.

Proof. From Theorem 2.15 the operator S^* takes the form $S^* = \lambda I_{\mathfrak{X}_{\mathcal{F}_2}} + W$ with $\lambda \in \mathbb{R}$ and $W : \mathfrak{X}_{\mathcal{F}_2} \to \mathfrak{X}_{\mathcal{F}_2}$ a strictly singular and weakly compact operator. We have to show that $\lambda = 0$.

The operator $W^* : \mathfrak{X}^*_{\mathcal{F}_2} \to \mathfrak{X}^*_{\mathcal{F}_2}$ is also weakly compact, while $W^* = S^{**} - \lambda I_{\mathfrak{X}^*_{\mathcal{F}_2}}$ which yields that $W^*((\mathfrak{X}_{\mathcal{F}_2})_*) \subset (\mathfrak{X}_{\mathcal{F}_2})_*$. These facts, in conjunction to the fact that $(\mathfrak{X}_{\mathcal{F}_2})_*$ contains no reflexive subspace (Theorem 4.7), imply that the restriction $W^*|(\mathfrak{X}_{\mathcal{F}_2})_*$ is strictly singular. Thus, since $\lambda I_{(\mathfrak{X}_{\mathcal{F}_2})_*} = S - W^*|(\mathfrak{X}_{\mathcal{F}_2})_*$ with both $S, W^*|(\mathfrak{X}_{\mathcal{F}_2})_*$ being strictly singular, we get that $\lambda = 0$. □

Corollary 4.10. Every bounded linear operator $T : (\mathfrak{X}_{\mathcal{F}_2})_* \to (\mathfrak{X}_{\mathcal{F}_2})_*$ takes the form $T = \lambda I + W$ with $\lambda \in \mathbb{R}$ and W a weakly compact operator.

Proof. We know from Theorem 2.16 that $T = \lambda I + W$ with W a strictly singular operator. Lemma 4.9 yields that W^* is also strictly singular. From Theorem 2.15 we get that W^* is weakly compact, hence W is weakly compact. □

Theorem 4.11. Let Z be a w^* closed subspace of $\mathfrak{X}_{\mathcal{F}_2}$ of infinite codimension such that for every $i = 1, 2, \dots$ we have that

$$\liminf_{k \in \Lambda_i} \operatorname{dist}(e_k, Z) = 0 \tag{7}$$

($(\Lambda_i)_{i \in \mathbb{N}}$ are the sets appearing in Definition 1.3). Then every infinite dimensional subspace of $\mathfrak{X}_{\mathcal{F}_2}/Z$ has nonseparable dual.

Proof. We denote by Q the quotient operator $Q : \mathfrak{X}_{\mathcal{F}_2} \to \mathfrak{X}_{\mathcal{F}_2}/Z$ and we recall that since Z is w^* closed, $Z_\perp$ 1-norms $\mathfrak{X}_{\mathcal{F}_2}/Z$. Let Y be a closed subspace of $\mathfrak{X}_{\mathcal{F}_2}$ with $Z \subset Y$ such that Y/Z is infinite dimensional; we shall show that $(Y/Z)^*$ is nonseparable.

For a given $j \in \mathbb{N}$ using Lemma 2.20 and our assumption (7) are able to construct a $(18, 4j-3, 1)$ attracting sequence $\chi = (x_k, x^*_k)_{k=1}^{n_{4j-3}}$ such that each one of the sums $\sum \operatorname{dist}(x_{2k-1}, Y)$, $\sum \operatorname{dist}(x^*_{2k-1}, Z_\perp)$, $\sum \|x_{2k-1}\|_{\mathcal{F}_2}$, $\sum \operatorname{dist}(x_{2k}, Z)$ is as small as we wish. Setting $d^1_\chi = \frac{m^2_{4j-3}}{n_{4j-3}} \sum\limits_{k=1}^{n_{4j-3}/2} x_{2k-1}$ we get that Qd_χ is almost equal to Qd^1_χ which almost belongs to Y/Z. Also F_χ almost belongs to $Z_\perp$, while $F_\chi(d_\chi) = \frac{1}{2}$ and $\|F_\chi - g_\chi\| \le \frac{1}{m_{4j-3}}$.

Using these estimates we are able to construct a dyadic tree $(\chi_a)_{a \in \mathcal{D}}$ of attracting sequences and a family $(j_a)_{a \in \mathcal{D}}$ of integers satisfying

(i) If $a <_{lex} \beta$ then $d_{\chi_a} < d_{\chi_\beta}$.

(ii) For every $\beta \in \mathcal{D}$, $\chi_\beta = \left(x^\beta_k, (x^\beta_k)^*\right)_{k=1}^{n_{4j_\beta-3}}$ is a $(18, 4j_\beta - 3, 1)$ attracting sequence with $\operatorname{dist}(F_{\chi_\beta}, Z_\perp) < \frac{1}{m_{4j_\beta-3}}$, $\operatorname{dist}(Qd_{\chi_\beta}, Y/Z) < \frac{1}{m_{4j_\beta-3}}$ and $\|x^\beta_{2k-1}\|_{\mathcal{F}_2} \le \frac{2}{m^2_{4j_\beta-3}}$ for $k = 1, \dots, n_{4j_\beta-3}/2$.

(iii) If $\beta \in \mathcal{D}$ with $\beta \ne \emptyset$ then $j_\beta = \sigma_{\mathcal{F}}((g_{\chi_a})_{a<\beta})$, while $j_\emptyset \in \Xi_1$ with $j_\emptyset \ge 3$.

For every $\beta \in \mathcal{D}$ we select $H_\beta \in Z_\perp$ with $\|H_\beta - F_{\chi_\beta}\| < \frac{1}{m_{4j_\beta-3}}$ and then for every branch b of $\mathcal{D}$ we denote by h_b the w^* limit of the series $\sum_{\beta\in b} H_\beta$.

Using the above estimates, and arguing similarly to the proof of Theorem 4.7, we obtain that $\{h_b|Y : \ b$ is a branch of $\mathcal{D}\}$ is a discrete family in $(Y/Z)^*$ and therefore $(Y/Z)^*$ is nonseparable. □

Remark 4.12. Actually it can be shown that the space $\mathfrak{X}_{\mathcal{F}_2}/Z$ is HJT.

Corollary 4.13. There exists a partition of the basis $(e_n^*)_{n\in\mathbb{N}}$ of $(\mathfrak{X}_{\mathcal{F}_2})_*$ into two sets $(e_n^*)_{n\in L_1}$, $(e_n^*)_{n\in L_2}$ such that setting $X_{L_1} = \overline{\text{span}}\{e_n^* : \ n \in L_1\}$, $X_{L_2} = \overline{\text{span}}\{e_n^* : \ n \in L_2\}$ both $X_{L_1}^*$, $X_{L_2}^*$ are HI with no reflexive subspace.

Proof. We choose $L_1 \in [\mathbb{N}]$ such that the sets $\Lambda_i \cap L_1$ and $\Lambda_i \setminus L_1$ are infinite for each i and we set $L_2 = \mathbb{N} \setminus L_1$. The spaces $X_{L_i} = \overline{\text{span}}\{e_n^* : \ n \in L_i\}$, $i = 1, 2$ satisfy the desired properties. Indeed, since $X_{L_1}^*$ is isometric to $\mathfrak{X}_{\mathcal{F}_2}/\overline{\text{span}}\{e_n : \ n \in L_2\}$, Theorem 4.11 yields that $X_{L_1}^*$ has no reflexive subspace while from Theorem 2.21 we get that it is an HI space. For $X_{L_2}^*$ the proof is completely analogous. □

5. The structure of $\mathfrak{X}_{\mathcal{F}_2}^*$ and a variant of $\mathfrak{X}_{\mathcal{F}_2}$

In the present section the structure of $\mathfrak{X}_{\mathcal{F}_2}^*$ is studied. This space is not HI since for every subspace Y of $(\mathfrak{X}_{\mathcal{F}_2})_*$ the space ℓ_2 embeds into Y^{**}. We also present a variant of $\mathfrak{X}_{\mathcal{F}_2}$, denoted $\mathfrak{X}_{\mathcal{F}_2'}$, such that $\mathfrak{X}_{\mathcal{F}_2'}^*/(\mathfrak{X}_{\mathcal{F}_2'})_*$ is isomorphic to $\ell_2(\Gamma)$ which yields some peculiar results on the structure of $\mathfrak{X}_{\mathcal{F}_2'}$ and $\mathfrak{X}_{\mathcal{F}_2'}^*$. Another variant of $\mathfrak{X}_{\mathcal{F}_2}$ yielding a HI dual not containing reflexive subspace is also discussed. It is well known that, in JT (James tree space) the quotient space JT^*/JT_* is isometric to $\ell_2(\Gamma)$. It seems unlikely to have the same property for $\mathfrak{X}_{\mathcal{F}_2}$. The main difficulty concerns the absence of biorthogonality between disjoint $\sigma_{\mathcal{F}}$ special functionals. However the next Proposition indicates that in some cases phenomena analogous to those in JT also occur.

Proposition 5.1. Let $(b_n^*)_n$ be a disjoint family of $\sigma_{\mathcal{F}}$ special functionals each one defined by an infinite special sequence $b_n = (f_1^n, \ldots, f_k^n, \ldots)$. Assume furthermore that for each (n, k) there exists a $(18, 4j_{(n,k)} - 3, 1)$ attracting sequence $\chi_{(n,k)} = (x_\ell^{(n,k)}, (x_\ell^{(n,k)})^*)_{\ell=1}^{n_{4j_{(n,k)}-3}}$, (Definition 1.15) with $\|x_{2\ell-1}^{(n,k)}\|_{\mathcal{F}_2} < \frac{1}{n^2_{4j_{(n,k)}-3}}$ and $f_n^k = g_{\chi_{(n,k)}}$, (Definition 4.4).

Then $(b_n^*)_n$ is equivalent to the standard ℓ_2-basis.

Let's provide a short description of the proof. We start with the following lemma:

Lemma 5.2. Let $\chi = (x_k, x_k^*)_{k=1}^{n_{4j-3}}$ be a $(18, 4j - 3, 1)$ attracting sequence such that $\|x_{2k-1}\|_{\mathcal{F}_2} < \frac{1}{n^2_{4j-3}}$ for all k. Then for every $\phi \in \mathcal{F}_2$ of the form $\phi = \sum_{i=1}^{d} a_i\phi_i$ with $j \notin \cup_{i=1}^{d} \text{ind}(\phi_i)$ we have that $|\phi(d_\chi)| < \frac{1}{m_{4j-3}}$. (Recall that $d_\chi = \frac{m^2_{4j-3}}{n_{4j-3}} \sum_{k=1}^{n_{4j-3}} (-1)^k x_k$, see Definition 4.4).

Proof. We set $d_\chi^1 = \frac{m_{4j-3}^2}{n_{4j-3}} \sum_{k=1}^{n_{4j-3}/2} x_{2k-1}$ and $d_\chi^2 = \frac{m_{4j-3}^2}{n_{4j-3}} \sum_{k=1}^{n_{4j-3}/2} x_{2k}$. From our assumption that $\|x_{2k-1}\|_{\mathcal{F}_2} < \frac{1}{n_{4j-3}^2}$ for every k, we get that $|\phi(d_\chi^1)| \leq \frac{m_{4j-3}^2}{n_{4j-3}} \frac{n_{4j-3}}{2} \frac{1}{n_{4j-3}^2}$.

If $f \in F_i$ for some $i < j$ we have that $|f(d_\chi^2)| \leq \frac{1}{m_{4i-3}^2} \frac{m_{4j-3}^2}{n_{4j-3}} \frac{n_{4i-3}}{2}$, while for $f \in F_i$ with $i > j$ we have that $|f(d_\chi^2)| \leq \frac{1}{m_{4i-3}^2} \frac{m_{4j-3}^2}{n_{4j-3}} \frac{n_{4j-3}}{2}$.

Therefore

$$\begin{aligned} |\phi(d_\chi)| &\leq |\phi(d_\chi^1)| + |\sum_{i=1}^d a_i \phi_i(d_\chi^2)| \leq |\phi(d_\chi^1)| + \sum_{i=1}^d |\phi_i(d_\chi^2)| \\ &\leq |\phi(d_\chi^1)| + \sum_{i<j} \sup\{|f(d_\chi^2)| : \ f \in F_i\} + \sum_{i>j} \sup\{|f(d_\chi^2)| : \ f \in F_i\} \\ &\leq \frac{m_{4j-3}^2}{2n_{4j-3}^2} + \frac{m_{4j-3}^2}{n_{4j-3}} \sum_{i<j} \frac{n_{4i-3}}{2m_{4i-3}^2} + \frac{m_{4j-3}^2}{2} \sum_{i>j} \frac{1}{m_{4i-3}^2} \\ &< \frac{1}{m_{4j-3}}. \end{aligned}$$

$\square$

The content of the above lemma is that each b^*, defined by an infinite $\sigma_{\mathcal{F}}$ special sequence b as in the previous proposition, is almost biorthogonal to any other $(b')^*$ which is disjoint from b.

Next we describe the main steps in the proof of Proposition 5.1.

Proof of Proposition 5.1: The proof follows the main lines of the proof of Lemma 11.3 of [AT1]. Given $(a_i)_{i=1}^d$ with $a_i \in \mathbb{Q}$ such that $\sum_{i=1}^d a_i^2 = 1$, we have that $\sum_{i=1}^d a_i b_i^* \in \mathcal{F}_2$ hence $\| \sum_{i=1}^d a_i b_i^* \| \leq 1$.

In order to complete the proof we shall show that

$$\frac{1}{1000} \leq \| \sum_{i=1}^d a_i b_i^* \| \tag{8}$$

which yields the desired result.

To establish (8), we choose $k \in \mathbb{N}$ with $\frac{(5n_{2k-1})^{\log_2(m_{2k})}}{n_{2k}} < \frac{\varepsilon}{4d}$ and then we choose $\{l_t^i : \ 1 \leq i \leq d, \ 1 \leq t \leq n_{2k}\}$ such that setting $x_{(t,i)} = d_{\chi_{(i,l_t^i)}}$ the following conditions are satisfied. First, the sequence $(x_{(t,i)})_{1 \leq i \leq d, \, 1 \leq t \leq n_{2k}}$ ordered lexicographically (i.e. $(t,i) <_{lex} (t',i')$ iff $t < t'$ or $t = t'$ and $i < i'$) is a $(144, \varepsilon)$ R.I.S. with associated sequence $4j'_{(t,i)} - 3 := 4j_{(i,l_t^i)} - 3$ while $m_{4j'_{(1,1)}-3} > \frac{2dn_{2k}}{\varepsilon}$.

We set $z_i = \frac{1}{n_{2k}} \sum_{t=1}^{n_{2k}} x_{(t,i)}$ for $i = 1, \ldots, d$. In order to prove (8) it is enough to show that

(i) $(\sum_{r=1}^d a_r b_r^*)(\sum_{i=1}^d a_i z_i) > \frac{1}{2} - \varepsilon$.

(ii) $\|\sum_{i=1}^{d} a_i z_i\| \leq 288$.

(i) is an easy consequence of Lemma 5.2. Indeed

$$\begin{aligned}(\sum_{r=1}^{d} a_r b_r^*)(\sum_{i=1}^{d} a_i z_i) &= \sum_{r=1}^{d} a_r^2 b_r^*(z_r) + \sum_{i=1}^{d}\sum_{r\neq i} a_r b_r^*(z_i)\\ &\geq \frac{1}{2} - \frac{1}{n_{2k}}\sum_{i=1}^{d}|a_i|\cdot|(\sum_{r\neq i} a_r b_r^*)(\sum_{t=1}^{n_{2k}} x_{(t,i)})|\\ &\geq \frac{1}{2} - \frac{1}{n_{2k}}\sum_{i=1}^{d}|a_i|(\sum_{t=1}^{n_{2k}}\frac{1}{4m_{4j'_{(t,i)}-3}})\\ &\geq \frac{1}{2} - \frac{1}{n_{2k}}\frac{2d}{m_{4j'_{(1,1)}-3}} > \frac{1}{2} - \varepsilon.\end{aligned}$$

For each (t,i) we set $k_{(t,i)} = \min\operatorname{supp} x_{(t,i)}$ we set and $s_i = \{k_{(t,i)} : t = 1,2,\ldots,n_{2k}\}$. We consider the set

$$\begin{aligned}\mathcal{H}_2 = \{e_n^* : n\in\mathbb{N}\} \quad\cup\quad & \Big\{\sum_{i=1}^{d}\sum_{j}\lambda_{i,j}s_{i,j}^* : \lambda_{i,j}\in\mathbb{Q},\ \sum_{i=1}^{d}\sum_{j}\lambda_{i,j}^2\leq 1, \text{ where}\\ & (s_{i,j})_j \text{ are disjoint subintervals of } s_i\Big\}\end{aligned}$$

and the norming set D' of space $T[\mathcal{H}_2,(\mathcal{A}_{5n_j},\frac{1}{m_j})_{j\in\mathbb{N}}]$.

We also set $\tilde{z}_i = \frac{1}{n_{2k}}\sum_{t=1}^{n_{2k}} e_{k_{(t,i)}}^*$ for $i = 1,\ldots,d$.

Claim. For every $f \in D_{\mathcal{F}_2}$ (where $D_{\mathcal{F}_2}$ is the norming set of the space $\mathfrak{X}_{\mathcal{F}_2}$) there exist an $h \in D'$ with nonnegative coordinates such that $|f(\sum_{i=1}^{d} a_i z_i)| \leq 288h(\sum_{i=1}^{d}|a_i|\tilde{z}_i) + \varepsilon$.

The proof of the above claim is obtained using similar methods to the proof of the basic inequality (Proposition A.5).

Arguing in a similar manner to the corresponding part of Lemma 11.3 of [AT1] we shall show that $h(\sum_{i=1}^{d}|a_i|\tilde{z}_i) \leq 1+\varepsilon$. We may assume that the functional h admits a tree $T_h = (h_a)_{a\in\mathcal{A}}$ (see Definition A.1) such that each h_a is either of type 0 (then $h_a \in \mathcal{H}_2$) or of type I, and moreover that the coordinates of each h_a are nonnegative. Let $(g_{a_s})_{s=1}^{s_0}$ be the functionals corresponding to the maximal elements of the tree $\mathcal{A}$. We denote by $\preceq$ the ordering of the tree $\mathcal{A}$. Let

$$A = \left\{ s\in\{1,2,\ldots,s_0\} : \prod_{\gamma\prec a_s}\frac{1}{w(h_\gamma)} \leq \frac{1}{m_{2k}} \right\}$$

$$B = \{1,2,\ldots,s_0\}\setminus A$$

and set $h_A = h|_{\bigcup_{s\in A}\operatorname{supp} g_{a_s}}$, $h_B = h|_{\bigcup_{s\in B}\operatorname{supp} g_{a_s}}$.

We have that $h_A(\tilde{z}_i) \leq \dfrac{1}{m_j}$ for each i thus

$$h_A(\sum_{i=1}^{d} |a_i|\tilde{z}_i) \leq \frac{1}{m_{2k}} \sum_{i=1}^{d} |a_i| \leq \frac{d}{m_j} < \frac{\varepsilon}{2}. \tag{9}$$

It remains to estimate the value $h_B(\sum\limits_{i=1}^{d} |a_i|\tilde{z}_i)$. We observe that

$$\sum_{i=1}^{d} |a_i|\tilde{z}_i = \sum_{t=1}^{n_{2k}} \frac{1}{n_{2k}} (\sum_{i=1}^{d} |a_i| e_{k_{(t,i)}}).$$

We set

$$\begin{aligned} E_1 &= \Big\{ t \in \{1,2,\ldots,n_{2k}\} : \text{ the set } \{k_{(t,1)}, k_{(t,2)}, \ldots, k_{(t,d)}\} \text{ is contained} \\ &\quad \text{in } \operatorname{ran} g_{a_s} \text{ for some } s \in B \text{ or does not intersect any } \operatorname{ran} g_{a_s},\ s \in B \Big\} \\ E_2 &= \{1,2,\ldots,n_{2k}\} \setminus E_1. \end{aligned}$$

For each $s = 1,2,\ldots,s_0$ set $\theta_s = \frac{1}{n_{2k}} \# \Big\{ t : \ \{l_t^1, l_t^2, \ldots, l_t^d\} \subset \operatorname{ran} g_{a_s} \Big\}$ and observe that $\sum\limits_{s\in B} \theta_s \leq 1$.

We first estimate the quantity $g_{a_s}(\sum\limits_{t\in E_1} \frac{1}{n_{2k}} (\sum\limits_{i=1}^{d} |a_i| e_{k_{(t,i)}}))$ for $s \in B$. Each g_{a_s} being in $\mathcal{H}_2$ takes the form $g_{a_s} = \sum\limits_i \sum\limits_j \lambda_{i,j} s_{i,j}^*$. For $1 \leq i' \leq d$ we get that $(\sum\limits_j \lambda_{i',j} s_{i',j}^*)(\sum\limits_{t\in E_1} \frac{1}{n_{2k}} (\sum\limits_{i=1}^{d} |a_i| e_{k_{(t,i)}})) |\tilde{a}_{i'}| (\sum\limits_j \lambda_{i',j} s_{i',j}^*)(\sum\limits_{t\in E_1} \frac{1}{n_{2k}} e_{k_{(t,i)}})$ $\leq |\tilde{a}_{i'}| (\max\limits_j \lambda_{i',j}) \theta_s$. Thus

$$\begin{aligned} g_{a_s}(\sum_{t\in E_1} \frac{1}{n_{2k}} (\sum_{i=1}^{d} |a_i| e_{k_(t,i)})) &\leq \theta_s \sum_{i=1}^{d} |\tilde{a}_i| \max_j \lambda_{i,j} \\ &\leq \theta_s (\sum_{i=1}^{d} \max_j \lambda_{i,j}^2)^{\frac{1}{2}} (\sum_{i=1}^{d} |\tilde{a}_i|^2)^{\frac{1}{2}} \leq \theta_s. \end{aligned}$$

Therefore

$$h_B(\sum_{t\in E_1} \frac{1}{n_{2k}} (\sum_{i=1}^{d} |a_i| e_{k_{(t,i)}})) \leq \sum_{s\in B} g_{a_s}(\sum_{t\in E_1} \frac{1}{n_{2k}} (\sum_{i=1}^{d} |a_i| e_{k_{(t,i)}})) \leq \sum_{s\in B} \theta_s \leq 1. \tag{10}$$

From the definition of the set E_2, the set $\{k_{(t,1)}, k_{(t,2)}, \ldots, k_{(t,d)}\}$, for each $t \in E_2$, intersects at least one but is not contained in any $\operatorname{ran} g_{a_s}$, $s \in B$. Also as in the proof of Lemma A.4 we get that $\#(B) \leq (5n_{2k-1})^{\log_2(m_{2k})}$. These yield that $\#(E_2) \leq 2(5n_{2k-1})^{\log_2(m_{2k})}$. Therefore from our choice of k we

derive that
(11)

$$h_B(\sum_{t\in E_2}\frac{1}{n_{2k}}(\sum_{i=1}^{d}|a_i|e_{k_{(t,i)}}))\leq(\sum_{t\in E_2}\frac{1}{n_{2k}})(\sum_{i=1}^{d}|a_i|)<\frac{2(5n_{2k-1})^{\log_2(m_{2k})}}{n_{2k}}<\frac{\varepsilon}{2}.$$

From (9),(10) and (11), we conclude that

$$\begin{aligned} h(\sum_{i=1}^{d}|a_i|e_{k_{(t,i)}}) \quad &\leq \quad h_A(\sum_{i=1}^{d}|a_i|\tilde{z}_i)+h_B(\sum_{t\in E_1}\frac{1}{n_{2k}}(\sum_{i=1}^{d}|a_i|e_{k_{(t,i)}})) \\ &+h_B(\sum_{t\in E_2}\frac{1}{n_{2k}}(\sum_{i=1}^{d}|a_i|e_{k_{(t,i)}}))\leq\frac{\varepsilon}{2}+1+\frac{\varepsilon}{2}=1+\varepsilon. \end{aligned}$$

□

As a consequence we obtain the following:

Theorem 5.3. For every infinite dimensional subspace Y of $(\mathfrak{X}_{\mathcal{F}_2})_*$, the space ℓ_2 is isomorphic to a subspace of Y^{**}.

A variant of $\mathfrak{X}_{\mathcal{F}_2}$

Next we shall indicate how we can obtain a space $\mathfrak{X}_{\mathcal{F}_2'}$ similar to $\mathfrak{X}_{\mathcal{F}_2}$ satisfying the additional property that $\mathfrak{X}^*_{\mathcal{F}_2'}/(\mathfrak{X}_{\mathcal{F}_2'})_*$ is isomorphic to $\ell^2(\Gamma)$. Notice that such a space has the following peculiar property:

Proposition 5.4. Granting that $\mathfrak{X}^*_{\mathcal{F}_2'}/(\mathfrak{X}_{\mathcal{F}_2'})_*$ is isomorphic to $\ell^2(\Gamma)$, every infinite dimensional w^*-closed subspace Z of $\mathfrak{X}^*_{\mathcal{F}_2'}$ is either nonseparable or isomorphic to ℓ_2.

Proof. Let $Q:\mathfrak{X}^*_{\mathcal{F}_2'}\to\mathfrak{X}^*_{\mathcal{F}_2'}/(\mathfrak{X}_{\mathcal{F}_2'})_*$ be the quotient map. There are two cases. If there exists a subspace $Z'\hookrightarrow Z$ of finite codimension such that $Q|_{Z'}$ is an isomorphism, then Z is isomorphic to ℓ_2. If not then there exists a normalized block sequence $(v_n)_{n\in\mathbb{N}}$ in $(\mathfrak{X}_{\mathcal{F}_2'})_*$ such that $\sum_{n=1}^{\infty}\mathrm{dist}(v_n,Z)<\frac{1}{3456}$. Setting $V=\overline{\mathrm{span}}\{v_n:\ n\in\mathbb{N}\}$ we observe that $\mathrm{dist}(S_V,Z)\leq\frac{1}{3456}$ hence, since Z is w^*-closed,

$$\mathrm{dist}(S_{\overline{V}^{w^*}},Z)\leq\frac{1}{3456}. \tag{12}$$

As in the proof of Theorem 4.7 we consider a James Tree structure $(w_a)_{a\in\mathcal{D}}$ in V such that the corresponding family $\{w_b:\ b\in[\mathcal{D}]\}$ satisfies the following properties:

(i) $\|w_b\|\leq 2$ for every $b\in[\mathcal{D}]$.
(ii) For $b\neq b'$ in $[\mathcal{D}]$ we have that $\|w_b-w_{b'}\|\geq\frac{1}{576}$.

The above (i) and (12) yield that for every $b\in[\mathcal{D}]$ there exists $z_b\in Z$ such that

$$\|z_b-w_b\|\leq\frac{1}{1728} \tag{13}$$

From (13) and the above (ii) we conclude that for $b\neq b'$ in $[\mathcal{D}]$ we have that $\|z_b-z_{b'}\|\geq\frac{1}{1728}$ which yields that Z is nonseparable. □

The following summarizes some of the properties of the space $\mathfrak{X}_{\mathcal{F}_2'}$.

Corollary 5.5. There exists a separable Banach space $\mathfrak{X}_{\mathcal{F}_2'}$ such that

(i) The space $\mathfrak{X}_{\mathcal{F}_2'}$ is HI and reflexively saturated.
(ii) Every quotient of $\mathfrak{X}_{\mathcal{F}_2'}$ has a further quotient isomorphic to ℓ_2.
(iii) Every quotient of $\mathfrak{X}_{\mathcal{F}_2'}$ either has nonseparable dual or it is isomorphic to ℓ_2.
(iv) There exists a quotient of $\mathfrak{X}_{\mathcal{F}_2'}$ not containing reflexive subspaces.

Before presenting the definition of the norming set $D_{\mathcal{F}_2'}$ let's explain our motivation. First we observe that Proposition 5.1 yields that for a sequence $(b_n^*)_n$ satisfying the assumptions, the sequence $([b_n^*])_n$ in the quotient space $W = \mathfrak{X}_{\mathcal{F}_2}^*/(\mathfrak{X}_{\mathcal{F}_2})_*$ is equivalent to the ℓ_2 basis. This in particular yields that W contains copies of $\ell_2(\Gamma)$ with $\#\Gamma$ equal to the continuum. Our intention is to define $\mathcal{F}_2' \subset \mathcal{F}_2$ and $D_{\mathcal{F}_2'} \subset D_{\mathcal{F}_2}$ such that every infinite $\sigma_{\mathcal{F}}$-special sequence $b = (f_1, f_2, \dots, f_n, \dots)$ satisfies the requirements of Proposition 5.1 with respect to the norm induced by the set $D_{\mathcal{F}_2'}$. Clearly if this is accomplished, then granting Proposition 5.1, the quotient $\mathfrak{X}_{\mathcal{F}_2'}^*/(\mathfrak{X}_{\mathcal{F}_2'})_*$ will be equivalent to $\ell_2(\Gamma)$.

The norm in the space $\mathfrak{X}_{\mathcal{F}_2'}$ is induced by a set $D_{\mathcal{F}_2'}$ which in turn, is recursively defined as $\cup_{n=0}^{\infty} D_n$. The key ingredient is that the ground set $\mathcal{F}_2'$, which is a subset of $\mathcal{F}_2$, is also defined inductively following the definition of D_n. Thus in each step we define the set S_n of the $\sigma_{\mathcal{F}}$-special sequences related to $\mathcal{F}_2$ and from this set, the set $\mathcal{F}_2^n$.

For $n = 0$, we set $S_0 = \emptyset$, $D_0 = \{\pm e_n^* : n \in \mathbb{N}\}$.

For $n = 1$ we set $S_1 = \cup_{j=1}^{\infty} F_j$, $\mathcal{F}_2^1$ is defined from S_1 and D_1 results from $D_0 \cup \mathcal{F}_2^1$ after applying the operations of Definition 1.2 and taking rational convex combinations.

Assume that S_n, $\mathcal{F}_2^n$, D_n have been defined such that every $\sigma_{\mathcal{F}}$ special sequence $(f_1, \dots, f_d)$ in S_n satisfies $d \leq n$. The $\sigma_{\mathcal{F}}$ special sequence $f_1, \dots, f_d$ in S_n is called $n+1$**-extendable** if for each $1 \leq i \leq d$ there exists a $(18, 4j_i - 3, 1, D_n, \mathcal{F}_2^n)$ attracting sequence $\chi_i = (x_k, x_k^*)_{k=1}^{4n_{j_i}-3}$, with $f_i = g_{\chi_i}$ (Definition 4.4). Here a $(18, 4j_i - 3, 1, D_n, \mathcal{F}_2^n)$ attracting sequence is defined as in Definition 1.15 where the norm of the underlying space is induced by the set D_n and moreover $\|x_{2k-1}\|_{D_n} \leq 18$ and $\|x_{2k-1}\|_{\mathcal{F}_2^n} \leq \frac{1}{n_{4j_i-3}^2}$.

Then we set $S_{n+1} = S_n \cup \{(f_1, \dots, f_d) : (f_1, \dots, f_{d-1}) \text{ is a } n+1\text{-extendable } \sigma_{\mathcal{F}} \text{ special sequence}\}$.

Next we define $\mathcal{F}_2^{n+1}$ from S_{n+1} in the usual manner and then D_{n+1} from $D_n \cup \mathcal{F}_2^{n+1}$ as before.

This completes the inductive definition. We set $\mathcal{F}_2' = \cup_n \mathcal{F}_2^n$ and $D_{\mathcal{F}_2'} = \cup_n D_n$.

It is easy to see that for every $b = (f_n)_n$ such that $b^* \in \overline{\mathcal{F}_2'}^{w^*}$ the sequence $(f_n)_n$ satisfies the properties of Proposition 5.1 and this yields that indeed $\mathfrak{X}_{\mathcal{F}_2'}^*/(\mathfrak{X}_{\mathcal{F}_2'})_*$ is isomorphic to $\ell^2(\Gamma)$.

6. A nonseparable HI space with no reflexive subspace

In this section we proceed to the construction of a nonseparable HI space containing no reflexive subspace. The general scheme we shall follow is similar to the one used for the definition of $\mathfrak{X}_{\mathcal{F}_2}$. However there are two major differences. The first concerns saturation methods. In the present construction we shall use the operations $(\mathcal{S}_{n_j}, \frac{1}{m_j})_j$ for appropriate sequences $(m_j)_j$, $(n_j)_j$. The James Tree space which will play the role of the ground space is also different from $JT_{\mathcal{F}_2}$. Indeed the ground set $\mathcal{F}'_s$ is built on a family $(F_j)_j$ which is related to the Schreier families $(\mathcal{S}_{n_{4j-3}})_j$. Furthermore in $\mathcal{F}'_s$ we connect the $\sigma_{\mathcal{F}}$ special functionals with the use of the Schreier operation instead of taking ℓ_2 sums as in $\mathcal{F}_2$. Finally, $\mathcal{F}'_s$ is defined recursively as we did in the previous variant $\mathfrak{X}_{\mathcal{F}'_2}$ of $\mathfrak{X}_{\mathcal{F}_2}$. The spaces $(\mathfrak{X}_{\mathcal{F}'_s})_*$, $\mathfrak{X}_{\mathcal{F}'_s}$ share the same properties with $(\mathfrak{X}_{\mathcal{F}'_2})_*$, $\mathfrak{X}_{\mathcal{F}'_2}$. The difference occurs between $\mathfrak{X}^*_{\mathcal{F}'_2}$ and $(\mathfrak{X}_{\mathcal{F}'_s})^*$. Indeed, as we have seen $(\mathfrak{X}_{\mathcal{F}'_2})^*/(\mathfrak{X}_{\mathcal{F}'_2})_*$ is isomorphic to $\ell_2(\Gamma)$, while as it will be shown $(\mathfrak{X}_{\mathcal{F}'_s})^*/(\mathfrak{X}_{\mathcal{F}'_s})_*$ is isomorphic to $c_0(\Gamma)$ with $\#\Gamma$ equal to the continuum. The later actually yields all the desired properties for $(\mathfrak{X}_{\mathcal{F}'_s})^*$. Namely it is HI and it does not contain any reflexive subspace.

We recall the definition of $(\mathcal{S}_n)_n$, the first infinite sequence of the Schreier families. The first Schreier family $\mathcal{S}_1$ is the following

$$\mathcal{S}_1 = \{F \subset \mathbb{N} : \ \#F \leq \min F\} \cup \{\emptyset\}.$$

For $n \geq 1$ the definition goes as follows

$$\mathcal{S}_{n+1} = \left\{ F = \bigcup_{i=1}^{d} F_i : \ F_i \in \mathcal{S}_n \ F_i < F_{i+1}, \text{ for all } i \text{ and } d \leq \min F_1 \right\}.$$

Each $\mathcal{S}_n$ is, as can be easily verified by induction, compact, hereditary and spreading.

A finite sequence $(E_1, E_2, \ldots, E_k)$ of successive subsets of $\mathbb{N}$ is said to be $\mathcal{S}_n$ admissible, $n \in \mathbb{N}$, if $\{\min E_i : i = 1, \ldots, k\} \in \mathcal{S}_n$. A finite sequence $(f_1, f_2, \ldots, f_k)$ of vectors in c_{00} is said to be $\mathcal{S}_n$ admissible if the sequence $(\operatorname{supp} f_1, \operatorname{supp} f_2, \ldots, \operatorname{supp} f_k)$ is $\mathcal{S}_n$ admissible.

We fix two sequences of integers $(m_j)_{j\in\mathbb{N}}$ and $(n_j)_{j\in\mathbb{N}}$ defined as follows:

- $m_1 = 2$ and $m_{j+1} = m_j^{m_j}$.
- $n_1 = 1$, and $n_{j+1} = 2^{2m_{j+1}} n_j$.

Definition 6.1. (basic special convex combinations) Let $\varepsilon > 0$ and $j \in \mathbb{N}$, $j > 1$. A convex combination $\sum_{k\in F} a_k e_k$ of the basis $(e_k)_{k\in\mathbb{N}}$ is said to be an (ε, j) basic special convex combination ((ε, j) B.S.C.C.) if

(1) $F \in \mathcal{S}_{n_j}$
(2) For every $P \in \mathcal{S}_{2\log_2(m_j)(n_{j-1}+1)}$ we have that $\sum_{k\in P} a_k < \varepsilon$.
(3) The sequence $(a_k)_{k\in F}$ is a non increasing sequence of positive reals.

Remark 6.2. The basic special convex combinations have been used implicitly in [AD], their exact definition was given in [AMT] while they have systematically studied in [AT1].

Definition 6.3. (special convex combinations) Let $\varepsilon > 0$, $j \in \mathbb{N}$ with $j > 1$ and let $(x_k)_{k\in\mathbb{N}}$ be a block sequence of the standard basis. A convex combination $\sum\limits_{k\in F} a_k x_k$ of the sequence $(x_k)_{k\in\mathbb{N}}$ is said to be an (ε, j) special convex combination ((ε, j) S.C.C.) of $(x_k)_{k\in\mathbb{N}}$ if $\sum\limits_{k\in F} a_k e_{t_k}$ (where $t_k = \min\operatorname{supp} x_k$ for each k) is an (ε, j) basic special convex combination.

Moreover, if $\sum\limits_{k\in F} a_k x_k$ is a S.C.C. in a Banach space $(X, \|\ \|)$ such that $\|x_k\| \leq 1$ for all k and $\|\sum\limits_{k\in F} a_k x_k\| \geq \dfrac{1}{2}$ we say that $\sum\limits_{k\in F} a_k x_k$ is a seminormalized (ε, j) special convex combination of $(x_k)_{k\in\mathbb{N}}$.

Definition 6.4. We set $F_0 = \{\pm e_n^* : \ n \in \mathbb{N}\}$ while for $j = 1, 2, \ldots$ we set $F_j = \{\frac{1}{m_{4j-3}^2} \sum\limits_{i\in I} \pm e_i^* : \ I \in \mathcal{S}_{n_{4j-3}}\} \cup \{0\}$. We also set $F = \bigcup\limits_{j=0}^{\infty} F_j$.

Let's observe that the sequence $\mathcal{F} = (F_j)_{j=0}^{\infty}$ is a JTG family. The $\sigma_{\mathcal{F}}$ special sequences corresponding to this family are defined exactly as in Definition 3.2.

Definition 6.5. (σ coding, special sequences and attractor sequences) Let $\mathbb{Q}_s$ denote the set of all finite sequences $(\phi_1, \phi_2, \ldots, \phi_d)$ such that $\phi_i \in c_{00}(\mathbb{N})$, $\phi_i \neq 0$ with $\phi_i(n) \in \mathbb{Q}$ for all i, n and $\phi_1 < \phi_2 < \cdots < \phi_d$. We fix a pair Ω_1, Ω_2 of disjoint infinite subsets of $\mathbb{N}$. From the fact that $\mathbb{Q}_s$ is countable we are able to define a Gowers-Maurey type injective coding function $\sigma : \mathbb{Q}_s \to \{2j : \ j \in \Omega_2\}$ such that $m_{\sigma(\phi_1,\phi_2,\ldots,\phi_d)} > \max\{\frac{1}{|\phi_i(e_l)|} : l \in \operatorname{supp}\phi_i, \ i = 1, \ldots, d\} \cdot \max\operatorname{supp}\phi_d$. Also, let $(\Lambda_i)_{i\in\mathbb{N}}$ be a sequence of pairwise disjoint infinite subsets of $\mathbb{N}$ with $\min\Lambda_i > m_i$.

(A) A finite sequence $(f_i)_{i=1}^d$ is said to be a $\mathcal{S}_{n_{4j-1}}$ **special sequence** provided that
  - (i) $(f_1, f_2, \ldots, f_d) \in \mathbb{Q}_s$ and $(f_1, f_2, \ldots, f_d)$ is a $\mathcal{S}_{n_{4j-1}}$ admissible sequence, $f_i \in D_G$ for $i = 1, 2, \ldots, n_{4j-1}$.
  - (ii) $w(f_1) = m_{2k}$ with $k \in \Omega_1$, $m_{2k}^{1/2} > n_{4j-1}$ and for each $1 \leq i < d$, $w(f_{i+1}) = m_{\sigma(f_1,\ldots,f_i)}$.

(B) A finite sequence $(f_i)_{i=1}^d$ is said to be a $\mathcal{S}_{n_{4j-3}}$ **attractor sequence** provided that
  - (i) $(f_1, f_2, \ldots, f_d) \in \mathbb{Q}_s$ and $(f_1, f_2, \ldots, f_d)$ is a $\mathcal{S}_{n_{4j-3}}$ admissible sequence.
  - (ii) $w(f_1) = m_{2k}$ with $k \in \Omega_1$, $m_{2k}^{1/2} > n_{4j-3}$ and $w(f_{2i+1}) = m_{\sigma(f_1,\ldots,f_{2i})}$ for each $1 \leq i < \frac{d}{2}$.
  - (iii) $f_{2i} = e_{l_{2i}}^*$ for some $l_{2i} \in \Lambda_{\sigma(f_1,\ldots,f_{2i-1})}$, for $i = 1, \ldots, \frac{d}{2}$.

Definition 6.6. (The space $\mathfrak{X}_{\mathcal{F}_s'}$) In order to define the norming set D of the space $\mathfrak{X}_{\mathcal{F}_s'}$ we shall inductively define four sequences of subsets of $c_{00}(\mathbb{N})$, denoted as $(K_n)_{n\in\mathbb{N}}$, $(\tau_n)_{n\in\mathbb{N}}$, $(G_n)_{n\in\mathbb{N}}$, $(D_n)_{n\in\mathbb{N}}$.

We set $K_0 = F$ ($K_0^0 = F$, $K_0^j = \emptyset$, $j = 1, 2, \ldots$), $G_0 = F$, $\tau_0 = \emptyset$ and $D_0 = \operatorname{conv}_{\mathbb{Q}}(F)$. Suppose that K_{n-1}, τ_{n-1}, G_{n-1} and D_{n-1} have been defined. The inductive properties of $(K_n)_{n\in\mathbb{N}}$, $(\tau_n)_{n\in\mathbb{N}}$, $(G_n)_{n\in\mathbb{N}}$, $(D_n)_{n\in\mathbb{N}}$ are included in the inductive definition. We set

$$K_n^{2j} = K_{n-1}^{2j} \cup \{\frac{1}{m_{2j}}\sum_{i=1}^{d} f_i : \ f_1 < \cdots < f_d \text{ is } \mathcal{S}_{n_{2j}} \text{ admissible}, \ f_i \in D_{n-1}\}$$

$$\begin{aligned} K_n^{4j-3} = K_{n-1}^{4j-3} \quad \cup \quad & \{\pm E(\frac{1}{m_{4j-3}}\sum_{i=1}^{d} f_i) : \ (f_1,\dots,f_d) \text{ is a } \mathcal{S}_{n_{4j-3}} \text{ attractor} \\ & \text{sequence}, \ f_i \in K_{n-1} \text{ and } E \text{ is an interval of } \mathbb{N}\} \end{aligned}$$

$$\begin{aligned} K_n^{4j-1} = K_{n-1}^{4j-1} \quad \cup \quad & \{\pm E(\frac{1}{m_{4j-1}}\sum_{i=1}^{d} f_i) : \ (f_1,\dots,f_d) \text{ is a } \mathcal{S}_{n_{4j-1}} \text{ special} \\ & \text{sequence}, \ f_i \in K_{n-1} \text{ and } E \text{ is an interval of } \mathbb{N}\} \end{aligned}$$

$$K_n^0 = F$$

.

We set $K_n = \bigcup_{j=0}^{\infty} K_n^j$.

In order to define τ_n we need the following definition.

Definition 6.7. ((D_{n-1},j) exact functionals) A functional $f \in F$ is said to be (D_{n-1},j) exact if $f \in F_j$ and there exists $x \in c_{00}(\mathbb{N})$ with $\|x\|_{D_{n-1}} \leq 1000$, $\mathrm{ran}(x) \subset \mathrm{ran}(f)$, $f(x) = 1$ such that for every $i \neq j$, we have that $\|x\|_{F_i} \leq \frac{1000}{m_{4i-3}^2}$ if $i < j$ while $\|x\|_{F_i} \leq 1000\frac{m_{4j-3}^2}{m_{4i-3}^2}$ if $i > j$.

We set

$$\begin{aligned} \tau_n \quad = \quad & \{\pm E(\sum_{i=1}^{d} \phi_i) : \ d \leq n, E \text{ is an interval}, (\phi_i)_{i=1}^d \text{ is } \sigma_{\mathcal{F}} \text{ special} \\ & \text{and each } \phi_i \text{ is } (D_{n-1}, \mathrm{ind}(\phi_i)) \text{ exact}\}. \end{aligned}$$

We recall that for $\Phi = \pm E(\sum_{i=1}^{d} \phi_i) \in \tau_n$, $\mathrm{ind}(\Phi) = \{\mathrm{ind}(\phi_i) : \ E \cap \mathrm{ran}\,\phi_i \neq \emptyset\}$. We set

$$\begin{aligned} G_n \quad = \quad & \{\sum_{i=1}^{d} \varepsilon_i \Phi_i : \ \Phi_i \in \tau_n, \ \varepsilon_i \in \{-1,1\}, \ \min\operatorname{supp}\Phi_i \geq d, \\ & (\mathrm{ind}(\Phi_i))_{i=1}^d \text{ are pairwise disjoint}\} \end{aligned}$$

We set $D_n = \mathrm{conv}_{\mathbb{Q}}(K_n \cup G_n \cup D_{n-1})$.

We finally set $D = \bigcup_{n=0}^{\infty} D_n$. We also set $\tau = \bigcup_{n=0}^{\infty} \tau_n$, $\mathcal{F}_s' = \bigcup_{n=0}^{\infty} G_n$, $K = \bigcup_{n=0}^{\infty} K_n$. We set $K^j = \bigcup_{n=1}^{\infty} K_n^j$ for $j = 1, 2, \ldots$. For $f \in K^j$ we write $w(f) = m_j$. We notice that $w(f)$ is not necessarily uniquely determined.

We also need the following definition.

Definition 6.8. (**(D,j) exact functionals**) A functional $f \in F$ is said to be (D,j) exact if $f \in F_j$ and there exists $x \in c_{00}(\mathbb{N})$ with $(\|x\|_D =)\|x\| \leq 1000$, $\mathrm{ran}(x) \subset \mathrm{ran}(f)$, $f(x) = 1$ such that for every $i \neq j$, we have that $\|x\|_{F_i} \leq \frac{1000}{m_{4i-3}^2}$ if $i < j$ while $\|x\|_{F_i} \leq 1000\frac{m_{4j-3}^2}{m_{4i-3}^2}$ if $i > j$.

Remarks 6.9. (i) If the functional ϕ is (D_n, j) exact then it is also (D_k, j) exact for all $k \leq n$.

(ii) Let $(\phi_i)_{i\in\mathbb{N}}$ be a $\sigma_{\mathcal{F}}$ special sequence such that each ϕ_i is $(D, \mathrm{ind}(\phi_i))$ exact. Then each ϕ_i is $(D_n, \mathrm{ind}(\phi_i))$ exact for all n and $\sum\limits_{i=1}^{d} \phi_i \in \tau_n \subset \tau$ for all $n \geq d$. It follows that $\sum\limits_{i=1}^{\infty} \phi_i \in \overline{\tau}^{w^*} \subset \overline{\mathcal{F}'_s}^{w^*} \subset \overline{D}^{w^*} = B_{\mathfrak{X}^*_{\mathcal{F}'_s}}$.

(iii) Let $(\phi_i)_{i\in\mathbb{N}}$ be a $\sigma_{\mathcal{F}}$ special sequence such that $\sum\limits_{i=1}^{d} \phi_i \in \tau$ for all d. In this case we call the $\sigma_{\mathcal{F}}$ special sequence $(\phi_i)_{i\in\mathbb{N}}$ survivor and the functional $\Phi = \sum\limits_{i=1}^{\infty} \phi_i$ a survivor $\sigma_{\mathcal{F}}$ special functional. Then each ϕ_i is (D_n, j_i) exact (where $j_i = \mathrm{ind}(\phi_i)$) for all n, thus for each n there exists $x_{i,n}$ with $\|x_{i,n}\|_{D_n} \leq 1000$, $\mathrm{ran}(x_{i,n}) \subset \mathrm{ran}(\phi_i)$, $\phi_i(x_{i,n}) = 1$ and such that for every $k \neq j_i$, we have that $\|x_{i,n}\|_{F_k} \leq \frac{1000}{m_{4k-3}^2}$ if $k < j$ while $\|x_{i,n}\|_{F_k} \leq 1000\frac{m_{4j-3}^2}{m_{4k-3}^2}$ if $k > j$. Taking a subsequence of $(x_{i,n})_{n\in\mathbb{N}}$ norm converging to some x_i it is easily checked that $\|x_i\| \leq 1000$, $\phi_i(x_i) = 1$ while $\|x_i\|_{F_k} \leq \frac{1000}{m_{4k-3}^2}$ for $k < j$ and $\|x_i\|_{F_k} \leq 1000\frac{m_{4j-3}^2}{m_{4k-3}^2}$ for $k > j$.

A sequence $(x_i)_{i\in\mathbb{N}}$ satisfying the above property is called a sequence witnessing that the $\sigma_{\mathcal{F}}$ special sequence $(\phi_i)_{i\in\mathbb{N}}$ (or the special functional $\Phi = \sum\limits_{i=1}^{\infty} \phi_i$) is survivor.

Lemma 6.10. The norming set D of the space $\mathfrak{X}_{\mathcal{F}'_s}$ is the minimal subset of $c_{00}(\mathbb{N})$ satisfying the following conditions:

(i) $\mathcal{F}'_s \subset D$.

(ii) D is closed in the $(\mathcal{S}_{n_{2j}}, \frac{1}{m_{2j}})$ operations.

(iii) D is closed in the $(\mathcal{S}_{n_{4j-1}}, \frac{1}{m_{4j-1}})$ operations on $\mathcal{S}_{n_{4j-1}}$ special sequences.

(iv) D is closed in the $(\mathcal{S}_{n_{4j-3}}, \frac{1}{m_{4j-3}})$ operations on $\mathcal{S}_{n_{4j-3}}$ special sequences.

(v) D is symmetric, closed in the restrictions of its elements on intervals of $\mathbb{N}$ and rationally convex.

It is easily proved that the Schauder basis $(e_n)_{n\in\mathbb{N}}$ of the space $\mathfrak{X}_{\mathcal{F}'_s}$ is boundedly complete and that $\mathfrak{X}_{\mathcal{F}'_s}$ is an asymptotic ℓ_1 space. Since the space $JT_{\mathcal{F}'_s}$ is c_0 saturated (see Remark B.16 where we use the notation $JT_{\mathcal{F}_{\tau,s}}$ for such a space) we get the following.

Proposition 6.11. The identity operator $I : \mathfrak{X}_{\mathcal{F}'_s} \to JT_{\mathcal{F}_s}$ is strictly singular.

Remark 6.12. Applying the methods of [AT1] and taking into account that the identity operator $I : \mathfrak{X}_{\mathcal{F}'_s} \to JT_{\mathcal{F}_s}$ is strictly singular we may prove the following. For every $\varepsilon > 0$ and $j > 1$ every block subspace of $\mathfrak{X}_{\mathcal{F}'_s}$ contains a vector x which is a seminormalized (ε, j) S.C.C. with $\|x\|_{\mathcal{F}'_s} < \varepsilon$.

Definition 6.13. (exact pairs in $\mathfrak{X}_{\mathcal{F}'_s}$) A pair (x, f) with $x \in c_{00}$ and $f \in K$ is said to be a $(12, j, \theta)$ exact pair, where $j \in \mathbb{N}$, if the following conditions are satisfied:

(i) $1 \le \|x\| \le 12$, $f(x) = \theta$ and $\mathrm{ran}(f) = \mathrm{ran}(x)$.

(ii) For every $g \in K$ with $w(g) = m_i$ and $i < j$, we have that $|g(x)| \le 24/m_i$.

(iii) For every sequence $(\phi_i)_i$ in K with $m_j < w(\phi_1) < w(\phi_2) < \cdots$ we have that $\sum\limits_i |\phi_i(x)| \le 12/m_j$.

Proposition 6.14. For every $j \in \mathbb{N}$, $\varepsilon > 0$ and every block subspace Z of $\mathfrak{X}_{\mathcal{F}'_s}$, there exists a $(12, 2j, 1)$ exact pair (z, f) with $z \in Z$ and $\|z\|_{\mathcal{F}'_s} < \varepsilon$.

Proof. Since the identity operator $I : \mathfrak{X}_{\mathcal{F}'_s} \to JT_{\mathcal{F}_s}$ is strictly singular we may assume, passing to a block subspace of Z, that $\|z\|_{\mathcal{F}'_s} \le \frac{\varepsilon}{12}\|z\|$ for every $z \in Z$.

Let $(x_k)_{k\in\mathbb{N}}$ be a block sequence in Z such that $(x_k)_{k\in\mathbb{N}}$ is a $(2, \frac{1}{m_{2j}})$ R.I.S. and each x_k is a seminormalized $(\frac{1}{m_{j_k}}, j_k)$ S.C.C. Passing to a subsequence we may assume that $(b^*(x_k))_{k\in\mathbb{N}}$ converges for every $\sigma_{\mathcal{F}}$ branch b. We set $z_k = x_{2k-1} - x_{2k}$. Then $(z_k)_{k\in\mathbb{N}}$ is a $(4, \frac{1}{m_{2j}})$ R.I.S. such that $b^*(z_k) \to 0$ for every branch b.

We recall that each $g \in \mathcal{F}'_s$ has the form $g = \sum\limits_{i=1}^{d} \varepsilon_i \Phi_i^*$ with $\varepsilon_i \in \{-1, 1\}$, $(\Phi_i^*)_{i=1}^d \in \tau$ with $\min\operatorname{supp} x_i^* \ge d$ and $(\mathrm{ind}(x_i^*))_{i=1}^d$ pairwise disjoint. We may assume, replacing $(z_k)_{k\in\mathbb{N}}$ by an appropriate subsequence, that for every $g \in G$ we have that the set $\{\min\operatorname{supp} z_k : \ |g(z_k)| > \frac{1}{m_{2j}}\}$ belongs to $\mathcal{S}_2$, the second Schreier family.

It follows now from Proposition 6.2 of [AT1] that if $z = \sum_{k\in F} a_k z_k$ is a $(1/m_{2j}^2, 2j)$ special convex combination of $(z_k)_{k\in\mathbb{N}}$ and f is of the form $f = 1/m_{2j} \sum_{k\in F} f_k$ where $f_k \in K$ with $f_k(z_k) = 1$ and $\mathrm{ran}(f_k) = \mathrm{ran}(z_k)$ then (z, f) is the desired $(12, 2j, 1)$ exact pair. $\square$

Definition 6.15. (dependent sequences and attracting sequences in $\mathfrak{X}_{\mathcal{F}'_s}$)

(A) A double sequence $(x_k, x_k^*)_{k=1}^d$ is said to be a $(C, 4j-1, \theta)$ dependent sequence (for $C > 1$, $j \in \mathbb{N}$, and $0 \le \theta \le 1$) if there exists a sequence $(2j_k)_{k=1}^d$ of even integers such that the following conditions are fulfilled:

  (i) $(x_k^*)_{k=1}^d$ is a $\mathcal{S}_{n_{4j-1}}$ special sequence with $w(x_k^*) = m_{2j_k}$ for each k.

  (ii) Each (x_k, x_k^*) is a $(C, 2j_k, \theta)$ exact pair.

  (iii) Setting $t_k = \min\operatorname{supp} x_k$, we have that $t_1 > m_{2j}$ and $\{t_1, \ldots, t_d\}$ is a maximal element of $\mathcal{S}_{n_{4j-1}}$. (Observe, for later use, that

Remark 3.18 of [AT1] yields that there exist $(a_k)_{k=1}^d$ such that $\sum_{k=1}^d a_k e_{t_k}$ is a $(\frac{1}{m_{4j-1}^2}, 4j-1)$ basic special convex combination).

(B) A double sequence $(x_k, x_k^*)_{k=1}^d$ is said to be a $(C, 4j-3, \theta)$ attracting sequence (for $C > 1$, $j \in \mathbb{N}$, and $0 \le \theta \le 1$) if there exists a sequence $(2j_k)_{k=1}^d$ of even integers such that the following conditions are fulfilled:
   (i) $(x_k^*)_{k=1}^d$ is a $\mathcal{S}_{n_{4j-3}}$ attractor sequence with $w(x_{2k-1}^*) = m_{2j_{2k-1}}$ and $x_{2k}^* = e_{l_{2k}}^*$ where $l_{2k} \in \Lambda_{2j_{2k}}$ for all $k \le d/2$.
   (ii) $x_{2k} = e_{l_{2k}}$.
   (iii) Setting $t_k = \min\operatorname{supp} x_k$, we have that $t_1 > m_{2j}$ and $\{t_1, \dots, t_d\}$ is a maximal element of $\mathcal{S}_{n_{4j-3}}$. (Observe that Remark 3.18 of [AT1] yields that there exist $(a_k)_{k=1}^d$ such that $\sum_{k=1}^d a_k e_{t_k}$ is a $(\frac{1}{m_{4j-3}^2}, 4j-3)$ basic special convex combination).
   (iv) Each (x_{2k-1}, x_{2k-1}^*) is a $(C, 2j_{2k-1}, \theta)$ exact pair.

Proposition 6.16. The space $\mathfrak{X}_{\mathcal{F}_s'}$ is reflexively saturated and Hereditarily Indecomposable.

Proof. The proof that $\mathfrak{X}_{\mathcal{F}_s'}$ is reflexively saturated is a consequence of the fact that the identity operator $I : \mathfrak{X}_{\mathcal{F}_s'} \to JT_{\mathcal{F}_s}$ is strictly singular and its proof is identical to that of Proposition 1.21.

In order to show that the space $\mathfrak{X}_{\mathcal{F}_s'}$ is Hereditarily Indecomposable we consider a pair of block subspaces Y and Z and $\delta > 0$. We choose j such that $m_{4j-1} > \frac{192}{\delta}$.

Using Proposition 6.14 we may choose a $(12, 4j-1, 1)$ dependent sequence $(x_k, x_k^*)_{k=1}^d$ such that $\|x_{2k-1}\|_{\mathcal{F}_s} < \frac{2}{m_{4j-1}^2}$ for all k while $x_{2k-1} \in Y$ if k is odd and $x_{2k-1} \in Z$ if k is even. ¿From the observation in Definition 6.15(A)(iii) there exist $(a_k)_{k=1}^d$ such that $\sum_{k=1}^d a_k e_{t_k}$ is a $(\frac{1}{m_{4j-1}^2}, 4j-1)$ basic special convex combination (where $t_k = \min\operatorname{supp} x_k$). A variant of Proposition 1.17 (i) in terms of the space $\mathfrak{X}_{\mathcal{F}_s'}$ (using Proposition 6.2 of [AT1]) yields that $\|\sum_{k=1}^d (-1)^{k+1} a_k x_k\| \le \frac{96}{m_{4j-1}^2}$. On the other hand the functional $f = \frac{1}{m_{4j-1}} \sum_{k=1}^d x_k^*$ belongs to the norming set D of the space $\mathfrak{X}_{\mathcal{F}_s'}$ and estimating $f(\sum_k a_k x_k)$ we get that $\|\sum_{k=1}^d a_k x_k\| \ge \frac{1}{m_{2j-1}}$.

Setting $y = \sum_{k \text{ odd}} a_k x_k$ and $z = \sum_{k \text{ even}} a_k x_k$ we have that $y \in Y$ and $z \in Z$ while from the above inequalities we get that $\|y - z\| \le \delta \|y + z\|$. Therefore $\mathfrak{X}_{\mathcal{F}_s'}$ is a Hereditarily Indecomposable space. $\square$

Proposition 6.17. The dual space $\mathfrak{X}_{\mathcal{F}_s'}^*$ is the norm closed linear span of the w^* closure of $\mathcal{F}_s'$ i.e.

$$\mathfrak{X}_{\mathcal{F}_s'}^* = \overline{\operatorname{span}}(\overline{\mathcal{F}_s'}^{w^*}).$$

Proof. Assume the contrary. Then using arguments similar to those of the proof of Proposition 1.19 we may choose a $x^* \in \mathfrak{X}^*_{\mathcal{F}'_s}$ with $\|x^*\| = 1$, and a block sequence $(x_k)_{k\in\mathbb{N}}$ in $\mathfrak{X}_{\mathcal{F}'_s}$ with $x^*(x_k) > 1$ and $\|x_k\| \leq 2$ such that $x_k \xrightarrow{w} 0$ in $JT_{\mathcal{F}_s}$. Observe that the action of x^* ensures that every convex combination of $(x_k)_{k\in\mathbb{N}}$ has norm greater than 1.

We may choose a convex block sequence $(y_k)_{k\in\mathbb{N}}$ of $(x_k)_{k\in\mathbb{N}}$ with $\|y_k\|_{\mathcal{F}_s} < \frac{\varepsilon}{2}$ where $\varepsilon = \frac{1}{m_4}$. We select a block sequence $(z_k)_{k\in\mathbb{N}}$ of $(y_k)_{k\in\mathbb{N}}$ such that each z_k is a convex combination of $(y_k)_{k\in\mathbb{N}}$ and such that $(z_k)_{k\in\mathbb{N}}$ is $(4,\varepsilon)$ R.I.S. This is possible if we consider each z_k to be a $(\frac{1}{m_{i_k}}, i_k)$ S.C.C. and $m_{i_{k+1}}\varepsilon > \max\operatorname{supp} z_k$ for an appropriate increasing sequence of integers $(i_k)_{k\in\mathbb{N}}$. We then consider $x = \sum a_k z_k$, an $(\varepsilon, 4)$ S.C.C. of $(z_k)_{k\in\mathbb{N}}$. A variant of Proposition 6.2(1a) of [AT1] yields that $\|x\| \leq \frac{20}{m_4} < 1$. On the other hand, since x is a convex combination of $(x_k)_{k\in\mathbb{N}}$ we get that $\|x\| > 1$, a contradiction. $\square$

Definition 6.18. Let $(x_n)_{n\in\mathbb{N}}$ be a bounded block sequence in $JT_{\mathcal{F}_s}$ and $\varepsilon > 0$. We say that $(x_n)_{n\in\mathbb{N}}$ is *ε-separated* if for every $\phi \in \cup_{j\in\mathbb{N}} F_j$

$$\#\{n : |\phi(x_n)| \geq \varepsilon\} \leq 1.$$

In addition, we say that $(x_n)_{n\in\mathbb{N}}$ is *separated* if for every $L \in [\mathbb{N}]$ and $\varepsilon > 0$ there exists an $M \in [L]$ such that $(x_n)_{n\in M}$ is ε-separated.

Lemma 6.19. Let $(x_n)_{n\in\mathbb{N}}$ be a weakly null separated sequence in $JT_{\mathcal{F}_s}$. Then for every $\varepsilon > 0$, there exists an $L \in [\mathbb{N}]$ such that for all $y^* \in \overline{\tau}^{w^*}$,

$$\#\{n \in L : |y^*(x_n)| \geq \varepsilon\} \leq 2.$$

The proof of the above lemma is similar to that of Lemma 3.12.

Lemma 6.20. Let $(z_k)_{k\in\mathbb{N}}$ be a block sequence in $\mathfrak{X}_{\mathcal{F}'_s}$ such that each z_k is a $(\frac{1}{m_{2j_k}}, 2j_k)$ special convex combination of a normalized block sequence, where $(j_k)_{k\in\mathbb{N}}$ is strictly increasing. Then the sequence $(z_k)_{k\in\mathbb{N}}$ is separated.

Proof. Given $\varepsilon > 0$ and $L \in [\mathbb{N}]$ we have to find an $M \in [L]$ such that for every $\phi \in \bigcup_{j\in N} F_j$ we have that $|\phi(z_k)| > \varepsilon$ for at most one $k \in M$. For simplicity in our notation we may assume, passing to a subsequence, that $\frac{1}{m_{2j_1}} < \varepsilon$ and $\max\operatorname{supp} z_{k-1} < \varepsilon m_{2j_k}$ for each k.

Now let $\phi \in \bigcup_{j\in N} F_j$. Then ϕ takes the form $\phi = \frac{1}{m^2_{4j-3}} \sum_{i\in I} \pm e^*_i$ with $I \in \mathcal{S}_{n_{4j-3}}$ for some j. Let k_0 such that $2j_{k_0} < 4j-3 < 2j_{k_0+1}$.

We have that $|\phi(z_k)| \leq \frac{1}{m^2_{4j-3}} \sum_{i\in I} |e^*_i(z_k)| < \frac{1}{m_{4j-3}} \frac{1}{m_{2j_{k_0}}} \#\operatorname{supp}(z_k) < \varepsilon$ for every $k < k_0$. Also for $k > k_0$ we get that $|\phi(z_k)| \leq \frac{2}{m_{2j_k}} < \varepsilon$. Thus the subsequence we have selected is ε-separated and this finishes the proof of the lemma. $\square$

Remark 6.21. Let's observe, for later use, that easy modifications of the previous proof yield that for a sequence $(z_k)_{k\in\mathbb{N}}$ as above the sequence $(z_{2k-1} - z_{2k})_{k\in\mathbb{N}}$ is also separated.

Lemma 6.22. Let $(z_k)_{k\in\mathbb{N}}$ be a weakly null separated sequence in $\mathfrak{X}_{\mathcal{F}'_s}$. Then for every $\varepsilon > 0$ there exists $L \in [\mathbb{N}]$ such that for every $\phi \in \mathcal{F}'_s$, $\{\min \operatorname{supp} z_k : k \in L,\ |\phi(z_k)| > \varepsilon\} \in \mathcal{S}_2$.

Proof. The proof is almost identical to the proof of Lemma 10.9 of [AT1]. For the sake of completeness we include the proof here.

Using Lemma 6.19 we construct a sequence $(L_k)_{k\in\mathbb{N}}$ of infinite subsets of the natural numbers such that the following conditions hold

(i) $\min \operatorname{supp} x_{l_1} \geq 3$.

(ii) $l_k = \min L_k \notin L_{k+1}$ and $L_{k+1} \subset L_k$ for each $k \in \mathbb{N}$.

(iii) For each $k \in \mathbb{N}$ if $p_k = \max \operatorname{supp} x_{l_k}$ then for every segment $s \in \overline{\mathcal{T}}^{w^*}$ we have that
$$\#\{n \in L_{k+1} :\ |s^*(x_n)| > \frac{\varepsilon}{p_k}\} \leq 2.$$

We set $L = \{l_1, l_2, l_3, \ldots\}$ and we claim that the set L satisfies the required condition.

Indeed, let $\phi = \sum_{i=1}^{d} \varepsilon_i s_i^* \in \mathcal{F}_s$ where segments $s_1, s_2, \ldots, s_d$ are in $\overline{\mathcal{T}}^{w^*}$, have pairwise disjoint sets of indices, $d \leq \min s_i$ and $\varepsilon_i \in \{-1, 1\}$ for each $i = 1, 2, \ldots, d$. We set
$$l_{i_0} = \min\{n \in L :\ \operatorname{supp} x_n \cap \operatorname{supp} \phi \neq \emptyset\}.$$
Observe that $d \leq p_{i_0}$. We set
$$F = \{n \in L :\ |\phi(x_n)| > \varepsilon\}.$$
We have that $F \subset \{l_{i_0}, l_{i_0+1}, l_{i_0+2}, \ldots\}$ thus $F \setminus \{l_{i_0}\} \subset L_{i_0+1}$. Also (iii) yields that for each $i = 1, 2, \ldots, d$ the set $F_i = \{n \in L_{i_0+1} :\ |s_i^*(x_n)| > \frac{\varepsilon}{p_{i_0}}\}$ has at most two elements.

We observe that
$$F \setminus \{l_{i_0}\} \subset \bigcup_{i=1}^{d} F_i.$$
Indeed if $n \in L_{i_0+1}$ and $n \notin \bigcup_{i=1}^{d} F_i$ then by our inductive construction $|s_i^*(x_n)| \leq \frac{\varepsilon}{p_{i_0}}$ for each $i = 1, 2, \ldots, d$ thus $|\sum_{i=1}^{d} \varepsilon_i s_i^*(x_n)| \leq d\frac{\varepsilon}{p_{i_0}}$ and it follows that $|\phi(x_n)| \leq \varepsilon$ therefore $n \notin F$.

We conclude that $\#(F \setminus \{l_{i_0}\}) \leq 2d$. Also $\min \operatorname{supp} x_n > p_{i_0} \geq d$ for each $n \in F \setminus \{l_{i_0}\}$ hence the set $\{\min \operatorname{supp} x_n :\ n \in F \setminus \{l_{i_0}\}\}$ is the union of two sets belonging to the first Schreier family $\mathcal{S}_1$. Since $\min \operatorname{supp} x_{l_{i_0}} \geq 3$ the set $\{\min \operatorname{supp} x_n :\ n \in F\}$ is the union of three sets of $\mathcal{S}_1$ and its minimum is greater or equal to 3. It follows that
$$\{\min \operatorname{supp} x_n :\ n \in F\} \in \mathcal{S}_2$$
which completes the proof of the Lemma. $\square$

Lemma 6.23. For every block subspace Z of $(\mathfrak{X}_{\mathcal{F}'_s})_*$ and and $j > 1$, $\varepsilon > 0$ there exists a $(60, 2j, 1)$ exact pair (z, z^*) such that $\mathrm{dist}(z^*, Z) < \varepsilon$ and $\|z\|_{\mathcal{F}'_s} < \frac{2}{m_{2j}}$.

Proof. As in the proof of Theorem 8.3 of [AT1] we may select a block sequence $(z_k)_{k\in\mathbb{N}}$ and a sequence $(f_k)_{k\in\mathbb{N}}$ in D such that

(i) Each z_k is a $(\frac{1}{m_{2j_k}}, 2j_k)$ S.C.C. of a normalized block sequence and the sequence $(j_k)_k$ is strictly increasing.
(ii) $f_k(z_k) > \frac{1}{3}$ and $\mathrm{ran}\, f_k = \mathrm{ran}\, z_k$.
(iii) $\mathrm{dist}(f_k, Z) < \frac{1}{2^k}$.

We assume that the sequence $(z_k)_{k\in\mathbb{N}}$ is weakly Cauchy. Then the sequence $(z_{2k-1} - z_{2k})_{k\in\mathbb{N}}$ is weakly null while from Remark 6.21 this sequence is separated. From Lemma 6.22, for every $\varepsilon > 0$ there exists $L \in [\mathbb{N}]$ such that for every $\phi \in \mathcal{F}'_s$, $\{\min\mathrm{supp}(z_{2k-1} - z_{2k}) : k \in L, |\phi(z_{2k-1} - z_{2k})| > \varepsilon\} \in \mathcal{S}_2$.

The rest of the proof follows the argument of Theorem 8.3 of the Memoirs monograph [AT1]. □

Proposition 6.24. Every infinite dimensional subspace of $(\mathfrak{X}_{\mathcal{F}'_s})_*$ has non-separable second dual. In particular the space $(\mathfrak{X}_{\mathcal{F}'_s})_*$ contains no reflexive subspace.

Proof. Using Lemma 6.23 for every block subspace and every j we may select, similarly to Lemma 4.6, a $(60, 4j-3, 1)$ attracting sequence $\chi = (x_k, x_k^*)_{k=1}^d$ with $\sum_k \|x_{2k-1}\|_{\mathcal{F}'_s} < \frac{1}{m_{4j-3}^2}$ and $\sum_k \mathrm{dist}(x_{2k-1}^*, Z) < \frac{1}{m_{4j-3}}$. We recall at this point (see Definition 6.15(B)(iii)) that there exists a sequence $(a_k)_{k=1}^d$ such that $\sum_{k=1}^d a_k e_{t_k}$ is a $(\frac{1}{m_{4j-3}^2}, 4j-3)$ basic special convex combination (where $t_k = \min\mathrm{supp}\, x_k$).

We set $F_\chi = -\frac{1}{m_{4j-3}^2}\sum_k x_{2k-1}^*$ and $g_\chi = \frac{1}{m_{4j-3}^2}\sum_k x_{2k}^*$. From the fact that $(x_k^*)_{k=1}^d$ is a $\mathcal{S}_{n_{4j-3}}$ special sequence we have that $\|\frac{1}{m_{2j-1}}(x_1^* + x_2^* + \cdots + x_d^*)\| \le 1$ and since $g_\chi - F_\chi = \frac{1}{m_{2j-1}}(\frac{1}{m_{2j-1}}(x_1^* + x_2^* + \cdots + x_d^*))$ we get that $\|g_\chi - F_\chi\| \le \frac{1}{m_{2j-1}}$. We also have that $\mathrm{dist}(F_\chi, Z) < \frac{1}{m_{4j-3}}$.

Similarly to Proposition 7.5 of [AT1] and to Proposition 1.17 of the present paper, we may prove that $\|\sum_{k=1}^d (-1)^k a_k x_k\| \le \frac{300}{m_{4j-3}^2}$. Observe also that $g_\chi(\sum_{k=1}^d (-1)^k a_k x_k) = \frac{1}{m_{4j-3}^2}\sum_k a_{2k} \ge \frac{1}{3m_{4j-3}^2}$. From these inequalities it follows that there exists $1 \le \theta_\chi \le 900$ such that $g_\chi(d_\chi) = 1$ and $\|d_\chi\| \le 900$ where $d_\chi = 3\theta_\chi m_{4j-3}^2 \sum_{k=1}^d (-1)^k a_k x_k$. It is also easily checked that $\|d_\chi\|_{F_i} \le \frac{1000}{m_{4i-3}^2}$ for $i < j$ while $\|d_\chi\|_{F_i} \le 1000\frac{m_{4j-3}^2}{m_{4i-3}^2}$ if $i > j$. Thus the vector d_χ witnesses that the functional g_χ is (D, j) exact.

Using arguments similar to those of Theorem 4.7, for a given block subspace Z of $(\mathfrak{X}_{\mathcal{F}'_s})_*$ we construct a family $(\chi_a)_{a\in\mathcal{D}}$ ($\mathcal{D}$ is the dyadic tree) of dependent sequences with properties analogous to (i),(ii), (iii) of Theorem 4.7 and such that for every $a \in \mathcal{D}$ the functional g_{χ_a} is (D, j_a) exact. It follows that for every branch b of the dyadic tree the sum $\sum_{a \in b} g_{\chi_a}$ converges in the w^* topology to a survivor $\sigma_{\mathcal{F}}$ special functional $g_b \in B_{\mathfrak{X}^*_{\mathcal{F}'_s}}$ and there exists $z_b \in Z^{**}$ with $\|z_b - g_b\| < \frac{1}{1152}$. Then, as in the proof of Theorem 4.7 we obtain that Z^{**} is nonseparable. □

Proposition 6.25. The space $(\mathfrak{X}_{\mathcal{F}'_s})_*$ is Hereditarily Indecomposable.

Proof. Let Y, Z be a pair of block subspaces of $(\mathfrak{X}_{\mathcal{F}'_s})_*$. For every $j > 1$, using Lemma 6.23, we are able to construct a $(60, 4j-1, 1)$ dependent sequence $(x_k, x_k^*)_{k=1}^{n_{4j-1}}$ such that $\|x_k\|_{\mathcal{F}'_s} < \frac{1}{m^2_{4j-1}}$ while $\sum \mathrm{dist}(x^*_{2k-1}, Y) <$ and $\sum \mathrm{dist}(x^*_{2k}, Z) <$ for each k. From the observation in Definition 6.15(A)(iii) there exist $(a_k)_{k=1}^d$ such that $\sum_{k=1}^{d} a_k e_{t_k}$ is a $(\frac{1}{m^2_{4j-1}}, 4j-1)$ basic special convex combination. As in Proposition 6.16 we get that $\|\sum_{k=1}^{d}(-1)^{k+1} a_k x_k\| \leq \frac{480}{m^2_{4j-1}}$.

We set $h_Y = \frac{1}{m_{4j-1}} \sum_{k \text{ odd}} x_k^*$ and $h_Z = \frac{1}{m_{4j-1}} \sum_{k \text{ even}} x_k^*$. The functional $h_Y + h_Z = \frac{1}{m_{4j-1}} \sum_{k=1}^{d} x_k^*$ belongs to the norming set D hence $\|h_Y + h_Z\| \leq 1$. On the other hand the action of $h_Y - h_Z$ to the vector $\sum_{k=1}^{d}(-1)^{k+1} a_k x_k$ yields that $\|h_Y - h_Z\| \geq \frac{m_{4j-1}}{480}$.

From the above estimates and since $\mathrm{dist}(h_Y, Y) < 1$ and $\mathrm{dist}(h_Z, Z) < 1$ we may choose $f_Y \in Y$ and $f_Z \in Z$ with $\|f_Y - f_Z\| \geq (\frac{m_{4j-1}}{1440} - \frac{2}{3})\|f_Y + f_Z\|$. Since this can be done for arbitrary large j we obtain that $(\mathfrak{X}_{\mathcal{F}'_s})_*$ is Hereditarily Indecomposable. □

Proposition 6.26. The quotient space $\mathfrak{X}^*_{\mathcal{F}'_s}/(\mathfrak{X}_{\mathcal{F}'_s})_*$ is isomorphic to $c_0(\Gamma)$ where the set Γ coincides with the set of all survivor $\sigma_{\mathcal{F}}$ special sequences.

Proof. As follows from 6.17 the quotient space $\mathfrak{X}^*_{\mathcal{F}'_s}/(\mathfrak{X}_{\mathcal{F}'_s})_*$ is generated in norm by the classes of the elements of the set $\overline{\mathcal{F}'_s}^{w^*}$. Since clearly

$$\begin{aligned} \overline{\mathcal{F}'_s}^{w^*} \quad &= \quad F \cup \{\sum_{i=1}^{d} \varepsilon_i \Phi_i :\ \Phi_i \in \tau,\ \varepsilon_i \in \{-1, 1\},\ \min\operatorname{supp} \Phi_i \geq d, \\ &\qquad (\mathrm{ind}(\Phi_i))_{i=1}^d \text{ are pairwise disjoint}\} \end{aligned}$$

we get that

$$\mathfrak{X}^*_{\mathcal{F}'_s} = \overline{\mathrm{span}}(\{e_n^* :\ n \in \mathbb{N}\} \cup \{\Phi :\ \Phi \text{ is a survivor } \sigma_{\mathcal{F}} \text{ special functional}\})$$

Thus $\mathfrak{X}^*_{\mathcal{F}'_s}/(\mathfrak{X}_{\mathcal{F}'_s})_* = \overline{\mathrm{span}}\{\Phi + (\mathfrak{X}_{\mathcal{F}'_s})_* :\ \Phi \text{ is a survivor } \sigma_{\mathcal{F}} \text{ special functional}\}$ To prove that this space is isomorphic to $c_0(\Gamma)$ we shall show that for every choice $(\Phi)_{i=1}^d$ of pairwise different survivor $\sigma_{\mathcal{F}}$ special functionals and every choice of signs $(\varepsilon_i)_{i=1}^d$ we have that

$$\frac{1}{2000} \leq \| \sum_{i=1}^{d} \varepsilon_i (\Phi_i + (\mathfrak{X}_{\mathcal{F}_s'})_*) \| \leq 1. \tag{14}$$

We have that $\| \sum_{i=1}^{d} \varepsilon_i (\Phi_i + (\mathfrak{X}_{\mathcal{F}_s'})_*) \| = \lim_k \| \sum_{i=1}^{d} \varepsilon_i (E_k \Phi_i) \|$ where for each k, $E_k = \{k, k+1, \ldots\}$. The right part of inequality (14) follows directly, since for all but finite k the functional $E_k(\sum_{i=1}^{d} \varepsilon_i \Phi_i)$ belongs to $\overline{\mathcal{F}_s'}^{w^*} \subset B_{\mathfrak{X}^*_{\mathcal{F}_s'}}$.

For each $i = 1, \ldots, d$ let $\Phi_i = \sum_{l=1}^{\infty} \phi_l^i$ with $(\phi_l^i)_{l \in \mathbb{N}}$ a survivor $\sigma_{\mathcal{F}}$ special sequence and and let (x_l^i) be a sequence witnessing this fact (see Remark 6.9 (iii)). We choose k_0 such that $(\mathrm{ind}(E_{k_0}\Phi_i))_{i=1}^d$ are pairwise disjoint and $\min(\bigcup_{i=1}^{d} \mathrm{ind}(E_{k_0}\Phi_i)) = r_0$ with $m_{2r_0-1} > 10^{10}$.

Let $k \geq k_0$. We choose t such that $\mathrm{ran}(\phi_t^1) \subset E_k$ and let $\mathrm{ind}(\phi_t^1) = l_0$. We get that $\sum_{i=2}^{d} |\Phi_i(x_t^1)| \leq \sum_{r=r_0}^{l_0-1} \frac{1000}{m_{4r-3}^2} + \sum_{r=l_0+1}^{\infty} \frac{1000 m_{2l_0-1}^2}{m_{4r-3}^2} < \frac{1}{2}$. We thus get that

$$\|E_k(\sum_{i=1}^{d} \varepsilon_i \Phi_i)\| \geq \frac{1}{1000}(\Phi_1(x_t^1) - \sum_{i=2}^{d} |\Phi_i(x_t^1)|) > \frac{1}{1000}(1 - \frac{1}{2}) = \frac{1}{2000}.$$

The proof of the proposition is complete. □

Theorem 6.27. There exists a Banach space $\mathfrak{X}_{\mathcal{F}_s'}$ satisfying the following properties:

(i) The space $\mathfrak{X}_{\mathcal{F}_s'}$ is an asymptotic ℓ_1 space with a boundedly complete Schauder basis $(e_n)_{n \in \mathbb{N}}$ and is Hereditarily Indecomposable and reflexively saturated.

(ii) The predual space $(\mathfrak{X}_{\mathcal{F}_s'})_* = \overline{\mathrm{span}}\{e_n^* : n \in \mathbb{N}\}$ is Hereditarily Indecomposable and each infinite dimensional subspace of $(\mathfrak{X}_{\mathcal{F}_s'})_*$ has nonseparable second dual. In particular the space $(\mathfrak{X}_{\mathcal{F}_s'})_*$ contains no reflexive subspace.

(iii) The dual space $\mathfrak{X}^*_{\mathcal{F}_s'}$ is nonseparable, Hereditarily Indecomposable and contains no reflexive subspace.

(iv) Every bounded linear operator $T : X \to X$ where $X = (\mathfrak{X}_{\mathcal{F}_s'})_*$ or $X = \mathfrak{X}_{\mathcal{F}_s'}$ or $X = \mathfrak{X}^*_{\mathcal{F}_s'}$ takes the form $T = \lambda I + W$ with W a weakly compact operator. In particular each $T : \mathfrak{X}^*_{\mathcal{F}_s'} \to \mathfrak{X}^*_{\mathcal{F}_s'}$ is of the form $T = Q^* + K$ with $Q : \mathfrak{X}_{\mathcal{F}_s'} \to \mathfrak{X}_{\mathcal{F}_s'}$ and K a compact operator, hence $T = \lambda I + R$ with R an operator with separable range.

Proof. As we have observed the Schauder basis $(e_n)_{n \in \mathbb{N}}$ of $\mathfrak{X}_{\mathcal{F}_s'}$ is boundedly complete and $\mathfrak{X}_{\mathcal{F}_s'}$ is asymptotic ℓ_1. In Proposition 6.16 we have shown that $\mathfrak{X}_{\mathcal{F}_s'}$ is reflexively saturated and Hereditarily Indecomposable. The facts that $(\mathfrak{X}_{\mathcal{F}_s'})_*$ is Hereditarily Indecomposable and that every subspace of it has nonseparable second dual have been shown in Proposition 6.24 and Proposition 6.25.

From the facts that the quotient space $\mathfrak{X}^*_{\mathcal{F}'_s}/(\mathfrak{X}_{\mathcal{F}'_s})_*$ is isomorphic to $c_0(\Gamma)$ and $(\mathfrak{X}_{\mathcal{F}'_s})_*$ is Hereditarily Indecomposable and taking into account that $\mathfrak{X}_{\mathcal{F}'_s}$, being a Hereditarily Indecomposable space, contains no isomorphic copy of ℓ_1 we get that the dual space $\mathfrak{X}^*_{\mathcal{F}'_s}$ is also Hereditarily Indecomposable (Corollary 1.5 of [AT1]). Since $\mathfrak{X}^*_{\mathcal{F}'_s}$ is Hereditarily Indecomposable and contains a subspace (which is $(\mathfrak{X}_{\mathcal{F}'_s})_*$ with no reflexive subspace we conclude that $\mathfrak{X}^*_{\mathcal{F}'_s}$ also does not have any reflexive subspace.

Using similar arguments to those of the proof of Theorems 2.15, 2.16 and Corollary 4.10 we may prove that every bounded linear operator $T:(\mathfrak{X}_{\mathcal{F}'_s})_*\to(\mathfrak{X}_{\mathcal{F}'_s})_*$ and every bounded linear operator $T:\mathfrak{X}_{\mathcal{F}'_s}\to\mathfrak{X}_{\mathcal{F}'_s}$ takes the form $T=\lambda I+W$ with W a strictly singular and weakly compact operator. Since $\mathfrak{X}_{\mathcal{F}'_s}$ contains no isomorphic copy of ℓ_1 and $\mathfrak{X}^{**}_{\mathcal{F}'_s}$ is isomorphic to $\mathfrak{X}_{\mathcal{F}'_s}\oplus\ell_1(\Gamma)$ Proposition 1.7 of [AT1] yields that every bounded linear operator $T:\mathfrak{X}^*_{\mathcal{F}'_s}\to\mathfrak{X}^*_{\mathcal{F}'_s}$ is of the form $T=Q^*+K$ with $Q:\mathfrak{X}_{\mathcal{F}'_s}\to\mathfrak{X}_{\mathcal{F}'_s}$ and K a compact operator. From the form of the operators of $\mathfrak{X}_{\mathcal{F}'_s}$ we have mentioned before we conclude that T takes the form $T=\lambda I+R$ with R a weakly compact operator and hence of separable range. $\square$

Remark 6.28. It is worth mentioning that the key ingredient to obtain $\mathfrak{X}^*_{\mathcal{F}'_s}/(\mathfrak{X}_{\mathcal{F}'_s})_*$ isomorphic to $c_0(\Gamma)$ which actually yields the HI property of $\mathfrak{X}^*_{\mathcal{F}'_s}$ is that in the ground set $\mathcal{F}'_s$ we connect the $\sigma_{\mathcal{F}}$ special functionals using the Schreier operation. This forces us to work with the saturation families $(\mathcal{S}_{n_j},\frac{1}{m_j})_j$ instead of $(\mathcal{A}_{n_j},\frac{1}{m_j})_j$. The reason for this is that working with $\mathcal{F}'_s$ built on $(F_j)_j$ with $F_j=\left\{\frac{1}{m^2_{4j-3}}\sum\limits_{i\in I}\pm e^*_i:\ \#(I)\le\frac{n_{4j-3}}{2}\right\}$ the extension with attractors of this ground set $\mathcal{F}'_s$ based on $(\mathcal{A}_{n_j},\frac{1}{m_j})_j$ is not strongly strictly singular.

However there exists an alternative way of connecting the $\sigma_{\mathcal{F}}$ special functionals lying between the Schreier operation and the ℓ_2 sums. This yields the James Tree space $JT_{\mathcal{F}_{2,s}}$ defined and studied in Appendix B. It is easy to check that the corresponding HI extension with attractors $\mathfrak{X}_{\mathcal{F}'_{2,s}}$ of $JT_{\mathcal{F}'_{2,s}}$ is a strictly singular one either we work on in the frame of $(\mathcal{A}_{n_j},\frac{1}{m_j})_j$ or of $(\mathcal{S}_{n_j},\frac{1}{m_j})_j$. It is open whether the corresponding space $\mathfrak{X}^*_{\mathcal{F}'_{2,s}}$ contains ℓ_2 or not. If it does not contain ℓ_2 then $\mathfrak{X}^*_{\mathcal{F}'_{2,s}}$ will be also a nonseparable HI space not containing any reflexive subspace with the additional property that $\mathfrak{X}^*_{\mathcal{F}'_{2,s}}/(\mathfrak{X}_{\mathcal{F}'_{2,s}})_*$ is isomorphic to $\ell_2(\Gamma)$.

7. A HJT space with unconditionally and reflexively saturated dual

This section concerns the definition of the space $\mathfrak{X}^{us}_{\mathcal{F}_2}$ namely a separable space with a boundedly complete basis which is reflexive and unconditionally saturated and its predual $(\mathfrak{X}^{us}_{\mathcal{F}_2})_*$ is HJT space hence it does not contain any reflexive subspace. This construction starts with the ground set $\mathcal{F}_2$ used in Section 4. In the extensions we use only attractors for which we eliminate a sufficient part of their conditional structure. The proof of the property that

$\mathfrak{X}^{us}_{\mathcal{F}_2}$ is unconditionally saturated follows the arguments of [AM],[AT2] while the HJT property of $(\mathfrak{X}^{us}_{\mathcal{F}_2})_*$ results from the remaining part of the conditional structure of the attractors. We additionally show that $(\mathfrak{X}^{us}_{\mathcal{F}_2})_*$ is HI.

Let $\mathbf{Q}$ be the set of all finitely supported scalar sequences with rational coordinates, of maximum modulus 1 and nonempty support. We set

$$\begin{aligned}\mathbf{Q_s} \quad = \quad & \{(x_1, f_1, \dots, x_n, f_n):\ x_i, f_i \in \mathbb{Q},\ i = 1, \dots, n \\ & \operatorname{ran}(x_i) \cup \operatorname{ran}(f_i) < \operatorname{ran}(x_{i+1}) \cup \operatorname{ran}(f_{i+1}),\ i = 1, \dots, n-1\}.\end{aligned}$$

For $\phi = (x_1, f_1, \dots, x_n, f_n) \in \mathbf{Q_s}$ and $l \le n$ we denote by ϕ_l the sequence $(x_1, f_1, \dots, x_l, f_l)$. We consider an injective coding function $\sigma : \mathbf{Q_s} \to \{2j : j \in \mathbb{N}\}$ such that for every $\phi = (x_1, f_1, \dots, x_n, f_n) \in \mathbf{Q_s}$

$$\begin{aligned}\sigma(x_1, f_1, \dots, x_{n-1}, f_{n-1}) < \sigma(x_1, f_1, \dots, x_n, f_n) \\ \text{and } \max\{\operatorname{ran}(x_n) \cup \operatorname{ran}(f_n)\} \le m^{\frac{1}{2}}_{\sigma(\phi)}.\end{aligned}$$

The norming set D^{us} of the space $\mathfrak{X}^{us}_{\mathcal{F}_2}$ will be defined as $D^{us} = \bigcup\limits_{n=0}^{\infty} D_n$ after defining inductively two sequences $(K_n)_{n=0}^{\infty}$, $(D_n)_{n=0}^{\infty}$ of subsets of $c_{00}(\mathbb{N})$ with $D_n = \operatorname{conv}_{\mathbb{Q}}(K_n)$.

Let $\mathcal{F}_2$ be the set defined in the beginning of the third section. We set

$$K_0 = \mathcal{F}_2 \quad \text{and} \quad D_0 = \operatorname{conv}_{\mathbb{Q}}(K_0).$$

Assume that K_{n-1} and D_{n-1} have been defined. Then for each $j \in \mathbb{N}$ we set

$$K^{2j}_n = K^{2j}_{n-1} \cup \{\frac{1}{m_{2j}} \sum_{i=1}^{d} f_i :\ f_1 < \dots < f_d,\ f_i \in D_{n-1},\ d \le n_{2j}\}.$$

For fixed $j \in \mathbb{N}$ we consider the collection of all sequences $\phi = (x_i, f_i)_{i=1}^{n_{4j-3}}$ satisfying the following conditions:

(i) $x_1 = e_{l_1}$ and $f_1 = e^*_{l_1}$ for some $l_1 \in \Lambda_{2j_1}$ where j_1 is an integer with $m^{1/2}_{2j_1} > n_{4j-3}$.

(ii) For $1 \le i \le n_{4j-3}/2$, $f_{2i} \in K^{\sigma(\phi_{2i-1})}_{n-1}$ and $\|x_{2i}\|_{K_{n-1}} \le \frac{18}{m_{\sigma(\phi_{2i-1})}}$.

(iii) For $1 \le i < n_{4j-3}/2$, $x_{2i+1} = e_{l_{2i+1}}$ and $f_{2i+1} = e^*_{l_{2i+1}}$ for some $l_{2i+1} \in \Lambda_{\sigma(\phi_{2i})}$.

For every ϕ satisfying (i),(ii) and (iii) we define the set

$$\begin{aligned}K^{4j-3}_{n,\phi} \quad = \quad & \big\{\frac{\pm 1}{m_{4j-3}} E\big(\sum_{i=1}^{n_{4j-3}/2} (\lambda_{f'_{2i}} f_{2i-1} + f'_{2i})\big) :\ E \text{ is an interval of } \mathbb{N}, \\ & f'_{2i} \in K^{\sigma(\phi_{2i-1})}_{n-1},\ \ \lambda_{f'_{2i}} = f'_{2i}(m_{\sigma(\phi_{2i-1})} x_{2i}), \\ & (x_{2i-1}, f_{2i-1}, x_{2i}, f'_{2i})_{i=1}^{n_{4j-3}/2} \in \mathbf{Q_s}\big\}.\end{aligned}$$

We define

$$K^{4j-3}_n = \cup\{K^{4j-3}_{n,\phi} :\ \phi \text{ satisfies conditions (i), (ii), (iii)}\} \cup K^{4j-3}_{n-1}$$

and we set

$$K_n = \bigcup_j (K_n^{2j} \cup K_n^{4j-3}) \text{ and } D_n = \mathrm{conv}_{\mathbb{Q}}(K_n).$$

We finally set

$$K^{us} = \bigcup_{n=0}^{\infty} K_n \qquad \text{and} \qquad D^{us} = \bigcup_{n=0}^{\infty} D_n$$

The space $\mathfrak{X}^{us}_{\mathcal{F}_2}$ is the completion of the space $(c_{00}, \| \, \|_{D^{us}})$ where

$$\|x\|_{D^{us}} = \sup\{f(x) : \ f \in D^{us}\}.$$

Using the same arguments as those in Proposition 4.1 we get the following.

Lemma 7.1. The identity operator $I : \mathfrak{X}^{us}_{\mathcal{F}_2} \to JT_{\mathcal{F}_2}$ is strongly strictly singular (Definition 2.1).

Definition 7.2. The sequence

$$\phi = (x_1, f_1, x_2, f_2, x_3, f_3, x_4, f_4, \dots, x_{n_{4j-3}}, f_{n_{4j-3}}) \in \mathbf{Q_s}$$

is said to be a n_{4j-3} attracting sequence provided that

(i) $x_1 = e_{l_1}$ and $f_1 = e^*_{l_1}$ for some $l_1 \in \Lambda_{2j_1}$ where j_1 is an integer with $m^{1/2}_{2j_1} > n_{4j-3}$.

(ii) For $1 \le i \le n_{4j-3}/2$, $(m_{\sigma(\phi_{2i-1})} x_{2i}, f_{2i})$ is a $(18, \sigma(\phi_{2i-1}), 1)$ exact pair (Definition 1.9) while $\sum\limits_{i=1}^{n_{4j-3}/2} \|x_{2i}\|_{\mathcal{F}_2} < \frac{1}{n_{4j-3}}$.

(iii) For $1 \le i < n_{4j-3}/2$, $x_{2i+1} = e_{l_{2i+1}}$ and $f_{2i+1} = e^*_{l_{2i+1}}$ for some $l_{2i+1} \in \Lambda_{\sigma(\phi_{2i})}$.

We consider the vectors d_ϕ in $\mathfrak{X}^{us}_{\mathcal{F}_2}$, and g_ϕ, F_ϕ in $(\mathfrak{X}^{us}_{\mathcal{F}_2})_*$ as they are defined in Definition 4.4. Let also notice, for later use, that the analogue of Lemma 4.5 remains valid.

Lemma 7.3. For every block subspace Z of the predual space $(\mathfrak{X}^{us}_{\mathcal{F}_2})_*$ and every $j \in N$ there exists a n_{4j-3} attracting sequence

$\phi = (x_1, f_1, x_2, f_2, \dots, x_{n_{4j-3}}, f_{n_{4j-3}})$ with $\sum\limits_{i=1}^{n_{4j-3}/2} \mathrm{dist}(f_{2i}, Z) < \frac{1}{m^2_{4j-3}}$.

Proof. Since the identity $I : \mathfrak{X}^{us}_{\mathcal{F}_2} \to JT_{\mathcal{F}_2}$ is strongly strictly singular (7.1), we may construct, using the analogue of Lemma 2.10 in terms of $\mathfrak{X}^{us}_{\mathcal{F}_2}$, the desired attracting sequence. □

Lemma 7.4. Let $\chi = (x_{2k}, x^*_{2k})_{k=1}^{n_{4j-3}/2}$ be a $(18, 4j-3, 1)$ attracting sequence such that $\sum\limits_{k=1}^{n_{4j-3}/2} \|x_{2k-1}\|_{\mathcal{F}_2} < \frac{1}{m^2_{4j-3}}$. Then for every branch b such that $j \notin \mathrm{ind}(b)$ we have that $|b^*(d_\chi)| < \frac{3}{m_{4j-3}}$.

Proof. Let $b = (f_1, f_2, f_3, \ldots)$ be a branch (i.e. $(f_i)_{i\in\mathbb{N}}$ is a $\sigma_{\mathcal{F}}$ special sequence). We recall that $d_\chi = \frac{m_{4j-3}^2}{n_{4j-3}} \sum_{k=1}^{n_{4j-3}} (-1)^k x_k$ and we set

$d_1 = -\frac{m_{4j-3}^2}{n_{4j-3}} \sum_{k=1}^{n_{4j-3}/2} x_{2k-1}$ and $d_2 = \frac{m_{4j-3}^2}{n_{4j-3}} \sum_{k=1}^{n_{4j-3}/2} x_{2k}$.

Our assumption $\sum_{k=1}^{n_{4j-3}/2} \|x_{2k-1}\|_{\mathcal{F}_2} < \frac{1}{m_{4j-3}^2}$ yields that

$$|b^*(d_1)| \leq \frac{m_{4j-3}^2}{n_{4j-3}} \sum_{k=1}^{n_{4j-3}/2} |b^*(x_{2k-1})| \leq \frac{m_{4j-3}^2}{n_{4j-3}} \sum_{k=1}^{n_{4j-3}/2} \|x_{2k-1}\|_{\mathcal{F}_2} < \frac{1}{n_{4j-3}}. \tag{15}$$

We decompose b^* as $b^* = x^* + y^*$ with $\mathrm{ind}(x^*) \subset \{1, \ldots, j-1\}$ and $\mathrm{ind}(y^*) \subset \{j+1, j+2, \ldots\}$. We recall that an $f \in F$ with $\inf(f) = l$ is of the form $f = \frac{1}{m_{4l-3}^2} \sum_{i\in\mathrm{supp}(f)} \pm e_i^*$ with $\mathrm{supp}(f) \leq n_{4l-3}/2$. Thus $\mathrm{supp}(x^*) \leq \frac{n_1}{2} + \frac{n_5}{2} + \cdots + \frac{n_{4j-7}}{2} < n_{4j-4}$. Hence

$$|x^*(d_2)| \leq \frac{m_{4j-3}^2}{n_{4j-3}} n_{4j-4} < \frac{1}{m_{4j-3}}. \tag{16}$$

On the other hand we have that $\|y^*\|_\infty \leq \frac{1}{m_{4j+1}}$, therefore

$$|y^*(d_2)| \leq \frac{1}{m_{4j+1}} \cdot \frac{m_{4j-3}^2}{n_{4j-3}} \cdot \frac{n_{4j-3}}{2} < \frac{1}{m_{4j-3}}. \tag{17}$$

From (15),(16) and (17) we obtain that $|b^*(d_\chi)| < \frac{3}{m_{4j-3}}$. □

Proposition 7.5. The space $(\mathfrak{X}_{\mathcal{F}_2}^{us})_*$ is Hereditarily Indecomposable.

Proof. Let Z_1, Z_2 be a pair of block subspaces in $(\mathfrak{X}_{\mathcal{F}_2}^{us})_*$ and let $0 < \delta < 1$.

We may inductively construct, using Lemma 7.3, a sequence $(\chi_r)_{r\in\mathbb{N}}$ such that the following conditions are satisfied.

(i) Each $\chi_r = (x_k^r, (x_k^r)^*)_{k=1}^{n_{4j_r-3}}$ is a $(18, 4j_r - 3, 1)$ attracting sequence with $\sum_{k=1}^{n_{2j_r-1}} \|x_{2k-1}^r\|_{\mathcal{F}_2} < \frac{1}{m_{4j_r-3}^2}$ and additionally $\mathrm{dist}(F_{\chi_r}, Z_1) < \frac{1}{m_{2j_r-1}}$ if r is odd, while $\mathrm{dist}(F_{\chi_r}, Z_2) < \frac{1}{m_{2j_r-1}}$ if r is even.

(ii) $(d_{\chi_r})_{r\in\mathbb{N}}$ is a block sequence.

(iii) For $r > 1$, $j_r = \sigma_{\mathcal{F}}(g_{\chi_1}, \ldots, g_{\chi_{r-1}})$.

Claim. The sequence $(d_{\chi_{2r-1}} - d_{\chi_{2r}})_{r\in\mathbb{N}}$ is a weakly null sequence in $\mathfrak{X}_{\mathcal{F}_2}$.

Proof of the claim. From the analogue of Proposition 1.19 the space $\mathfrak{X}_{\mathcal{F}_2}^*$ is the closed linear span of the pointwise closure $\overline{\mathcal{F}_2}^{w^*}$ of the set $\mathcal{F}_2$. From the observation after Theorem 3.7 we have that

$$\begin{aligned}\overline{\mathcal{F}_2}^{w^*} \quad = \quad & F_0 \cup \Big\{ \sum_{i=1}^\infty a_i x_i^* : \sum_{i=1}^\infty a_i^2 \leq 1,\ (x_i^*)_{i=1}^d \text{ are } \sigma_{\mathcal{F}} \text{ special functionals} \\ & \text{with } (\mathrm{ind}(x_i^*))_{i=1}^d \text{ pairwise disjoint } \min\mathrm{supp}\, x_i^* \geq d \Big\}.\end{aligned}$$

Thus it is enough to show that $b^*(d_{\chi_{2r-1}} - d_{\chi_{2r}}) \overset{r\to\infty}{\longrightarrow} 0$ for every branch b.

Let b be an arbitrary branch. If $b = (g_{\chi_1}, g_{\chi_2}, g_{\chi_3}, g_{\chi_4}, \ldots)$ from we obtain that $g(d_{\chi_{2r-1}} - d_{\chi_{2r}}) = g_{\chi_{2r-1}}(d_{\chi_{2r-1}}) - g_{\chi_{2r}}(d_{\chi_{2r}}) = \frac{1}{2} - \frac{1}{2} = 0$ for every r. If $b \neq (g_{\chi_1}, g_{\chi_2}, g_{\chi_3}, g_{\chi_4}, \ldots)$ the injectivity of the coding function $\sigma_{\mathcal{F}}$ yields that there exist $r_0 \in \mathbb{N}$ such that $j_r \notin \text{ind}(b^*)$ for all $r > 2r_0$. Hence, for $r > r_0$, Lemma 7.4 yields that $|b(d_{\chi_{2r-1}} - d_{\chi_{2r}})| \leq |b(d_{\chi_{2r-1}})| + |b(d_{\chi_{2r}})| < \frac{3}{m_{4j_{2r-1}-3}} + \frac{3}{m_{4j_{2r}-3}} < \frac{4}{m_{4j_{2r-1}-3}}$ and therefore $b^*(d_{\chi_{2r-1}} - d_{\chi_{2r}}) \overset{r\to\infty}{\longrightarrow} 0$.

The proof of the claim is complete. $\square$

It follows from the claim that there exists a convex combination of the sequence $(d_{\chi_{2r-1}} - d_{\chi_{2r}})_{r\in\mathbb{N}}$ with norm less than $\frac{\delta}{3}$; let $(a_r)_{r=1}^d$ be nonnegative reals with $\sum\limits_{r=1}^{d} a_r = 1$ such that $\|\sum\limits_{r=1}^{d} a_r(d_{\chi_{2r-1}} - d_{\chi_{2r}})\| < \frac{\delta}{3}$.

We set $g = \sum\limits_{r=1}^{2d} g_{\chi_r}$ and $g' = \sum\limits_{r=1}^{d}(g_{\chi_{2r-1}} - g_{\chi_{2r}})$. Since $g \in \mathcal{F}_2$ we have that $\|g\| \leq 1$. On the other hand

$$\begin{aligned}
\|g'\| &\geq \frac{g'\big(\sum\limits_{r=1}^{d} a_r(d_{\chi_{2r-1}} - d_{\chi_{2r}})\big)}{\|\sum\limits_{r=1}^{d} a_r(d_{\chi_{2r-1}} - d_{\chi_{2r}})\|} \\
&> \frac{\sum\limits_{r=1}^{d} a_r(g_{\chi_{2r-1}} - g_{\chi_{2r}})(d_{\chi_{2r-1}} - d_{\chi_{2r}})}{\frac{\delta}{3}} = \frac{\sum\limits_{r=1}^{d} a_r(\frac{1}{2}+\frac{1}{2})}{\frac{\delta}{3}} = \frac{3}{\delta}.
\end{aligned}$$

For each $r \leq 2d$ with r odd we select $z_r^* \in Z_1$ such that $\|z_r^* - F_{\chi_r}\| < \frac{1}{m_{4j_r-3}}$, while for r even we select $z_r^* \in Z_2$ such that $\|z_r^* - F_{\chi_r}\| < \frac{1}{m_{4j_r-3}}$. We set

$$F_1 = \sum_{r=1}^{d} z_{2r-1}^*(\in Z_1) \quad \text{and} \quad F_2 = \sum_{r=1}^{d} z_{2r}^*(\in Z_2).$$

From our choice of z_r^* and the analogue of Lemma 4.5 we get that

$$\sum_{r=1}^{2d}\|z_r - g_{\chi_r}\| \leq \sum_{r=1}^{2d}(\|z_r - F_{\chi_r}\| + \|F_{\chi_r} - g_{\chi_r}\|) \leq \sum_{r=1}^{2d}\Big(\frac{1}{m_{4j_r-3}} + \frac{1}{m_{4j_r-3}}\Big) < 1. \tag{18}$$

From (18) we obtain that $\|(F_1 + F_2) - g\| < 1$ and $\|(F_1 - F_2) - g'\| < 1$. Thus, the facts that $\|g\| \leq 1$ and $\|g'\| > \frac{3}{\delta}$ yield that $\|F_1 + F_2\| < 2$ and $\|F_1 - F_2\| > \frac{3}{\delta} - 1 > \frac{2}{\delta}$, therefore $\|F_1 + F_2\| < \delta\|F_1 + F_2\|$. The proof of the proposition is complete. $\square$

Proposition 7.6. The space $(\mathfrak{X}_{\mathcal{F}_2}^{us})_*$ is HJT. In particular the space $(\mathfrak{X}_{\mathcal{F}_2}^{us})_*$ contains no reflexive subspace and every infinite dimensional subspace Z of $(\mathfrak{X}_{\mathcal{F}_2}^{us})_*$ has nonseparable second dual Z^{**}.

Proof. The proof is identical to that of Theorem 4.7. $\square$

Theorem 7.7. The space $\mathfrak{X}_{\mathcal{F}_2}^{us}$ has the following properties

(i) Every subspace Y of $\mathfrak{X}^{us}_{\mathcal{F}_2}$ contains a further subspace Z which is reflexive and has an unconditional basis.

(ii) The predual $(\mathfrak{X}^{us}_{\mathcal{F}_2})_*$ of the space $\mathfrak{X}^{us}_{\mathcal{F}_2}$ is Hereditarily Indecomposable and has no reflexive subspace.

Proof. First, the identity operator $I : \mathfrak{X}^{us}_{\mathcal{F}_2} \to JT_{\mathcal{F}_2}$, being strongly strictly singular, is strictly singular (Proposition 2.3). Therefore the space $\mathfrak{X}^{us}_{\mathcal{F}_2}$ is reflexively saturated. Let Z be an arbitrary block subspace of the space $\mathfrak{X}^{us}_{\mathcal{F}_2}$. From the fact that $I : \mathfrak{X}^{us}_{\mathcal{F}_2} \to JT_{\mathcal{F}_2}$ is strictly singular we may choose a block sequence $(z_k)_{k\in\mathbb{N}}$ in Z with $\|z_k\| = 1$ and $\sum_{k=1}^{\infty} \|z_k\|_{\mathcal{F}_2} < \frac{1}{16}$. We may prove that $(z_k)_{k\in\mathbb{N}}$ is an unconditional basic sequence following the procedure used in the proof of Proposition 3.6 of [AT2].

The facts that the space $(\mathfrak{X}^{us}_{\mathcal{F}_2})_*$ is Hereditarily Indecomposable and has no reflexive subspace have been proved in Propositions 7.5 and 7.6. $\square$

Defining the norming set of the present section using $\mathcal{F}_s$ (instead of $\mathcal{F}_2$) in the first inductive step (namely in the definition of K_0) and using the saturation methods $(\mathcal{S}_{n_j}, \frac{1}{m_j})_j$ (instead of $(\mathcal{A}_{n_j}, \frac{1}{m_j})_j$) we produce a Banach space $\mathfrak{X}^{us}_{\mathcal{F}_s}$ which is unconditionally saturated while its predual and its dual share similar properties with the space $\mathfrak{X}_{\mathcal{F}_s}$ of Section 6. Namely we have the following.

Theorem 7.8. There exists a Banach space $\mathfrak{X}^{us}_{\mathcal{F}_s}$ with the properties:

(i) The predual $(\mathfrak{X}^{us}_{\mathcal{F}_s})_*$ of $\mathfrak{X}^{us}_{\mathcal{F}_s}$ is HI and every infinite dimensional subspace of $(\mathfrak{X}^{us}_{\mathcal{F}_s})_*$ has nonseparable second dual. In particular $(\mathfrak{X}^{us}_{\mathcal{F}_s})_*$ contains no reflexive subspace.

(ii) The space $\mathfrak{X}^{us}_{\mathcal{F}_s}$ is unconditionally and reflexively saturated.

(iii) The dual space $(\mathfrak{X}^{us}_{\mathcal{F}_s})^*$ is nonseparable HI and contains no reflexive subspace.

Appendix A. The auxiliary space and the basic inequality

The basic inequality is the main tool in providing upper bounds for the action of functionals on certain vectors of X_G. It has appeared in several variants in previous works like [AT1], [ALT], [ArTo]. In this section we present another variant which mainly concerns the case of strongly strictly singular extensions and in particular we provide the proof of Proposition 1.7 stated in Section 1. The proof of the present variant follows the same lines as the previous ones.

Definition A.1. The tree T_f of a functional $f \in W$. Let $f \in D$. By a tree of f (or tree corresponding to the analysis of f) we mean a finite family $T_f = (f_a)_{a\in\mathcal{A}}$ indexed by a finite tree $\mathcal{A}$ with a unique root $0 \in \mathcal{A}$ such that the following conditions are satisfied:

1. $f_0 = f$ and $f_a \in D$ for all $a \in \mathcal{A}$.
2. An $a \in \mathcal{A}$ is maximal if and only if $f_a \in G$.
3. For every $a \in \mathcal{A}$ which is not maximal, denoting by S_a the set of the immediate successors of a, exactly one of the following holds:

(a) $S_a = \{\beta_1, \ldots, \beta_d\}$ with $f_{\beta_1} < \cdots < f_{\beta_d}$ and there exists $j \in \mathbb{N}$ such that $d \le n_j$ and $f_a = \frac{1}{m_j} \sum\limits_{i=1}^{d} f_{\beta_i}$ (recall that in this case we say that f_a is of type I).

(b) $S_a = \{\beta_1, \ldots, \beta_d\}$ and there exists a family of positive rationals $\{r_{\beta_i} : \ i = 1, \ldots, d\}$ with $\sum\limits_{i=1}^{d} r_{\beta_i} = 1$ such that $f_a = \sum\limits_{i=1}^{d} r_{\beta_i} f_{\beta_i}$. Moreover for all $i = 1, \ldots, d$, $\operatorname{ran} f_{\beta_i} \subset \operatorname{ran} f_a$. (recall that in this case we say that f_a is of type II).

It is obvious that every $f \in D$ has a tree which is not necessarily unique.

Definition A.2. (The auxiliary space T_{j_0}) Let $j_0 > 1$ be fixed. We set $C_{j_0} = \{\sum\limits_{i \in F} \pm e_i^* : \ \#(F) \le n_{j_0 - 1}\}$.

The auxiliary space T_{j_0} is the completion of $(c_{00}(\mathbb{N}) \ , \| \ \|_{D_{j_0}})$ where the norming set D_{j_0} is defined to be the minimal subset of $c_{00}(\mathbb{N})$ which (i) Contains C_{j_0}. (ii) It is closed under $(\mathcal{A}_{5n_j}, \frac{1}{m_j})$ operations for all $j \in \mathbb{N}$. (iii) It is rationally convex.

Observe that the Schauder basis $(e_n)_{n \in \mathbb{N}}$ of T_{j_0} is 1-unconditional.

Remark A.3. Let D'_{j_0} be the minimal subset of $c_{00}(\mathbb{N})$ which (i) Contains C_{j_0}. (ii) Is closed under $(\mathcal{A}_{5n_j}, \frac{1}{m_j})$ operations for all $j \in \mathbb{N}$. We notice that each $f \in D'_{j_0}$ has a tree $(f_a)_{a \in \mathcal{A}}$ in which for $a \in \mathcal{A}$ which is not maximal, f is the result of an $(\mathcal{A}_{5n_j}, \frac{1}{m_j})$ operation (for some j) of the functionals $(f_\beta)_{\beta \in S_a}$.

It can be shown that D'_{j_0} is also a norming set for the space T_{j_0} and that for every $j \in \mathbb{N}$ we have that $\operatorname{conv}_{\mathbb{Q}}\{f \in D_{j_0} : w(f) = m_j\} = \operatorname{conv}_{\mathbb{Q}}\{f \in D'_{j_0} : w(f) = m_j\}$. For proofs of similar results in a different context we refer to [AT1] (Lemma 3.5).

Lemma A.4. Let $j_0 \in \mathbb{N}$ and $f \in D'_{j_0}$. Then for every family $k_1 < k_2 < \ldots < k_{n_{j_0}}$ we have that

$$|f(\frac{1}{n_{j_0}} \sum_{l=1}^{n_{j_0}} e_{k_l})| \le \begin{cases} \frac{2}{m_i \cdot m_{j_0}}, & \text{if } \ w(f) = m_i, \ i < j_0 \\ \frac{1}{m_i}, & \text{if } \ w(f) = m_i, \ i \ge j_0 \end{cases} \tag{19}$$

In particular $\|\frac{1}{n_{j_0}} \sum\limits_{l=1}^{n_{j_0}} e_{k_l}\|_{D_{j_0}} \le \frac{1}{m_{j_0}}$.

If we additionally assume that the functional f admits a tree $(f_\alpha)_{\alpha \in \mathcal{A}}$ such that $w(f_\alpha) \ne m_{j_0}$ for every $\alpha \in \mathcal{A}$, then we have that

$$|f(\frac{1}{n_{j_0}} \sum_{l=1}^{n_{j_0}} e_{k_l})| \le \begin{cases} \frac{2}{m_i \cdot m_{j_0}^2}, & \text{if } \ w(f) = m_i, \ i < j_0 \\ \frac{1}{m_i}, & \text{if } \ w(f) = m_i, \ i > j_0 \end{cases} \le \frac{1}{m_{j_0}^2}. \tag{20}$$

Proof. We first prove the following claim.

Claim. Let $h \in D'_{j_0}$. Then

(i) $\#\{k : \ |h(e_k)| > \frac{1}{m_{j_0}}\} < (5n_{j_0 - 1})^{\log_2(m_{j_0})}$.

(ii) If the functional h has a tree $(h_a)_{a\in\mathcal{A}}$ with $w(h_a)\neq m_{j_0}$ for each $a\in\mathcal{A}$ then
$\#\{k:\ |h(e_k)|>\frac{1}{m_{j_0}^2}\}<(5n_{j_0-1})^{2\log_2(m_{j_0})}$.

Proof of the claim. We shall prove only part (i) of the claim, as the proof of (ii) is similar. Let $(h_a)_{a\in\mathcal{A}}$ be a tree of h and let n be its height (i.e. the length of its maximal branch). We may assume that $|h(e_k)|>\frac{1}{m_{j_0}}$ for all $k\in\operatorname{supp}h$. Let $h=h_0,h_1,\dots,h_n$ be a maximal branch (then $h_n\in C_{j_0}$) and let $k\in\operatorname{supp}h_n$. Then $\frac{1}{m_{j_0}}<|h(e_k)|=\prod\limits_{l=0}^{n-1}\frac{1}{w(h_l)}\leq\frac{1}{2^n}$, hence $n\leq\log_2(m_{j_0})-1$.

On the other hand, since $|h(e_k)|>\frac{1}{m_{j_0}}$ for all $k\in\operatorname{supp}h$, each h_a with a non maximal is a result of an $(\mathcal{A}_{5n_j},\frac{1}{m_j})$ operation for $j\leq j_0-1$. An inductive argument yields that for $i\leq n$ the cardinality of the set $\{h_a:\ |a|=i\}$ is less or equal to $(5n_{j_0-1})^i$. The facts that $n\leq\log_2(m_{j_0})-1$ and that each element of $g\in C_{j_0}$ has $\#(\operatorname{supp}(g))\leq n_{j_0-1}$ yield that $\#(\operatorname{supp}(h))\leq n_{j_0-1}(5n_{j_0-1})^{\log_2(m_{j_0})-1}<(5n_{j_0-1})^{\log_2(m_{j_0})}$.

The proof of the claim is complete. $\square$

We pass to the proof of the lemma. The case $w(f)=m_i$, $i\geq j_0$ is straightforward. Let $f\in D'_{j_0}$ with $w(f)=m_i$, $i<j$. Then $f=\frac{1}{m_i}\sum\limits_{t=1}^{d}f_t$ where $f_1<\dots<f_d$ belong to D'_{j_0} and $d\leq n_i$.

For $t=1,\dots,d$ we set $H_t=\{k:\ |f_t(e_k)|>\frac{1}{m_{j_0}}\}$. Part (i) of the claim yields that $\#(H_t)<(5n_{j_0-1})^{\log_2(m_{j_0})}$. Thus, setting $H=\bigcup\limits_{t=1}^{d}H_t$, we get that $\#(H)<d(5n_{j_0-1})^{\log_2(m_{j_0})}\leq(5n_{j_0-1})^{\log_2(m_{j_0})+1}$. Therefore

$$\begin{aligned}|f(\frac{1}{n_{j_0}}\sum_{l=1}^{n_{j_0}}e_{k_l})| &\leq \frac{1}{m_i}\Big(\big|(\sum_{t=1}^{d}f_t)_{|H}(\frac{1}{n_{j_0}}\sum_{l=1}^{n_{j_0}}e_{k_l})\big|\Big)\\ &\quad+\frac{1}{m_i}\Big(\big|(\sum_{t=1}^{d}f_t)_{|(\mathbb{N}\setminus H)}(\frac{1}{n_{j_0}}\sum_{l=1}^{n_{j_0}}e_{k_l})\big|\Big)\\ &\leq \frac{1}{m_i}\#(H)\frac{1}{n_{j_0}}+\frac{1}{m_i}\frac{1}{m_{j_0}}<\frac{2}{m_im_{j_0}}.\end{aligned}$$

The second part is proved similarly by using part (ii) of the claim. $\square$

Proposition A.5. (The basic inequality) Let $(x_k)_{k\in\mathbb{N}}$ be a (C,ε) R.I.S. in X_G and $j_0>1$ such that for every $g\in G$ the set $\{k:\ |g(x_k)|>\varepsilon\}$ has cardinality at most n_{j_0-1}. Let $(\lambda_k)_{k\in\mathbb{N}}\in c_{00}$ be a sequence of scalars. Then for every $f\in D$ of type I we can find g_1, such that either $g_1=h_1$ or $g_1=e_t^*+h_1$ with $t\notin\operatorname{supp}h_1$ where $h_1\in\operatorname{conv}_{\mathbb{Q}}\{h\in D'_{j_0}:\ w(h)=w(f)\}$ and $g_2\in c_{00}(\mathbb{N})$ with $\|g_2\|_\infty\leq\varepsilon$ with g_1,g_2 having nonnegative coordinates and such that

$$|f(\sum\lambda_kx_k)|\leq C(g_1+g_2)(\sum|\lambda_k|e_k). \tag{21}$$

If we additionally assume that for every $h \in D$ with $w(h) = m_{j_0}$ and every interval E of the natural numbers we have that

$$|h(\sum_{k\in E} \lambda_k x_k)| \leq C(\max_{k\in E} |\lambda_k| + \varepsilon \sum_{k\in E} |\lambda_k|) \tag{22}$$

then, if $w(f) \neq m_{j_0}$, h_1 may be selected satisfying additionally the following property: $h_1 = \sum r_l \tilde{h}_l$ with $r_l \in \mathbb{Q}^+$, $\sum r_l = 1$ and for each l the functional $\tilde{h}_l$ belongs to D'_{j_0} with $w(\tilde{h}_l) = w(f)$ and admits a tree $T_{\tilde{h}_l} = (f^l_a)_{a\in\mathcal{C}_l}$ with $w(f^l_a) \neq m_{j_0}$ for all $a \in \mathcal{C}_l$.

Proof. The proof in the general case (where (22) is not assumed) and in the special case (where we assume (22)) is actually the same. We shall give the proof only in the special case. The proof in the general case arises by omitting any reference to distinguishing cases whether a functional has weight m_{j_0} or not and treating the functionals with $w(f) = m_{j_0}$ as for any other j.

We fix a tree $T_f = (f_a)_{a\in\mathcal{A}}$ of f. Before passing to the proof we adopt some useful notation and state two lemmas. Their proofs can be found in [AT1] (Lemmas 4.4 and 4.5).

Definition A.6. For each $k \in \mathbb{N}$ we define the set $\mathcal{A}_k$ as follows:

$$\begin{aligned} \mathcal{A}_k \quad = \quad \Big\{ & a \in \mathcal{A} \text{ such that } f_a \text{ is not of type } II \text{ and} \\ & (i) \;\; \operatorname{ran} f_a \cap \operatorname{ran} x_k \neq \emptyset \\ & (ii) \;\; \forall\, \gamma < a \text{ if } f_\gamma \text{ is of type } I \text{ then } w(f_\gamma) \neq m_{j_0} \\ & (iii) \;\; \forall\, \beta \leq a \quad \text{if } \beta \in S_\gamma \text{ and } f_\gamma \text{ is of type } I \\ & \;\; \text{then } \operatorname{ran} f_\beta \cap \operatorname{ran} x_k = \operatorname{ran} f_\gamma \cap \operatorname{ran} x_k \\ & (iv) \;\; \text{if } w(f_a) \neq m_{j_0} \text{ then for all } \beta \in S_a \\ & \;\; \operatorname{ran} f_\beta \cap \operatorname{ran} x_k \subsetneqq \operatorname{ran} f_a \cap \operatorname{ran} x_k \Big\} \end{aligned}$$

The next lemma describes the properties of the set $\mathcal{A}_k$.

Lemma A.7. For every $k \in \mathbb{N}$ we have the following:

(i) If $a \in \mathcal{A}$ and f_a is of type II then $a \notin \mathcal{A}_k$.
(Hence $\mathcal{A}_k \subset \{a \in \mathcal{A} : \; f_a \text{ is of type } I \text{ or } f_a \in G\}$.)

(ii) If $a \in \mathcal{A}_k$, then for every $\beta < a$ if f_β is of type I then $w(f_\beta) \neq m_{j_0}$.

(iii) If $\mathcal{A}_k$ is not a singleton then its members are incomparable members of the tree $\mathcal{A}$. Moreover if a_1, a_2 are two different elements of $\mathcal{A}_k$ and β is the (necessarily uniquely determined) maximal element of $\mathcal{A}$ satisfying $\beta < a_1$ and $\beta < a_2$ then f_β is of type II.

(iv) If $a \in \mathcal{A}$ is such that $\operatorname{supp} f_a \cap \operatorname{ran} x_k \neq \emptyset$ and $\gamma \notin \mathcal{A}_k$ for all $\gamma < a$ then there exists $\beta \in \mathcal{A}_k$ with $a \leq \beta$. In particular if $\operatorname{supp} f \cap \operatorname{ran} x_k \neq \emptyset$ then $\mathcal{A}_k \neq \emptyset$.

Definition A.8. For every $a \in \mathcal{A}$ we define $D_a = \bigcup_{\beta \geq a} \{k : \; \beta \in \mathcal{A}_k\}$.

Lemma A.9. According to the notation above we have the following:

(i) If $\operatorname{supp} f \cap \operatorname{ran} x_k \neq \emptyset$ then $k \in D_0$ (recall that 0 denotes the unique root of $\mathcal{A}$ and $f = f_0$). Hence $f(\sum \lambda_k x_k) = f(\sum_{k\in D_0} \lambda_k x_k)$.

(ii) If f_a is of type I with $w(f_a) = m_{j_0}$ then D_a is an interval of $\mathbb{N}$.

(iii) If f_a is of type I with $w(f_a) \neq m_{j_0}$ then

$$\left\{ \{k\} : \; k \in D_a \setminus \bigcup_{\beta \in S_a} D_\beta \right\} \cup \{D_\beta : \; \beta \in S_a\}$$

is a family of successive subsets of $\mathbb{N}$. Moreover for every $k \in D_a \setminus \bigcup_{\beta \in S_a} D_\beta$ (i.e. for k such that $a \in \mathcal{A}_k$) such that $\operatorname{supp} f_a \cap \operatorname{ran} x_k \neq \emptyset$ there exists a $\beta \in S_a$ such that either $\min \operatorname{supp} x_k \leq \max \operatorname{supp} f_\beta < \max \operatorname{supp} x_k$ or $\min \operatorname{supp} x_k < \min \operatorname{supp} f_\beta \leq \max \operatorname{supp} x_k$.

(iv) If f_a is of type II, $\beta \in S_a$ and $k \in D_a \setminus D_\beta$ then $\operatorname{supp} f_\beta \cap \operatorname{ran} x_k = \emptyset$ and hence $f_\beta(x_k) = 0$.

Recall that we have fixed a tree $(f_a)_{a \in \mathcal{A}}$ for the given f. We construct two families $(g_a^1)_{a \in \mathcal{A}}$ and $(g_a^2)_{a \in \mathcal{A}}$ such that the following conditions are fulfilled.

(i) For every $a \in \mathcal{A}$ such that f_a is not of type II, $g_a^1 = h_a$ or $g_a^1 = e_{k_a}^* + h_a$ with $t_a \notin \operatorname{supp} h_a$, where $h_a \in \operatorname{conv}_{\mathbb{Q}}(D'_{j_0})$ and $g_a^2 \in c_{00}(\mathbb{N})$ with $\|g_a^2\|_\infty \leq \varepsilon$.

(ii) For every $a \in \mathcal{A}$, $\operatorname{supp} g_a^1 \subset D_a$ and $\operatorname{supp} g_a^2 \subset D_a$ and the functionals g_a^1, g_a^2 have nonnegative coordinates.

(iii) For $a \in \mathcal{A}$ with $f_a \in G$ and $D_a \neq \emptyset$ we have that $g_a^1 \in C_{j_0}$.

(iv) For f_a of type II with $f = \sum_{\beta \in S_a} r_\beta f_\beta$ (where $r_\beta \in \mathbb{Q}^+$ for every $\beta \in S_a$ and $\sum_{\beta \in S_a} r_\beta = 1$) we have $g_a^1 = \sum_{\beta \in S_a} r_\beta g_\beta^1$ and $g_a^2 = \sum_{\beta \in S_a} r_\beta g_\beta^2$.

(v) For f_a of type I with $w(f) = m_{j_0}$ we have $g_a^1 = e_{k_a}^*$ where $k_a \in D_a$ is such that $|\lambda_{k_a}| = \max_{k \in D_a} |\lambda_k|$ and $g_a^2 = \sum_{k \in D_a} \varepsilon e_k^*$.

(vi) For f_a of type I with $w(f) = m_j$ for $j \neq j_0$ we have $g_a^1 = h_a$ or $g_a^1 = e_{k_a}^* + h_a$ with $h_a \in \operatorname{conv}_{\mathbb{Q}}\{h \in D'_{j_0} : \; w(h) = m_j\}$ and $k_a \notin \operatorname{supp} h_a$.

(vii) For every $a \in \mathcal{A}$ the following inequality holds:

$$|f_a(\sum_{k \in D_a} \lambda_k x_k)| \leq C(g_a^1 + g_a^2)(\sum_{k \in D_a} |\lambda_k| e_k).$$

When the construction of $(g_a^1)_{a \in \mathcal{A}}$ and $(g_a^2)_{a \in \mathcal{A}}$ has been accomplished, we set $g_1 = g_0^1$ and $g_2 = g_0^2$ (where 0 is the root of $\mathcal{A}$ and $f = f_0$) and we observe that these are the desired functionals. To show that such $(g_a^1)_{a \in \mathcal{A}}$ and $(g_a^2)_{a \in \mathcal{A}}$ exist we use finite induction starting with $a \in \mathcal{A}$ which are maximal and in the general inductive step we assume that g_β^1, g_β^2 have been defined for all $\beta > a$ satisfying the inductive assumptions and we define g_a^1 and g_a^2.

$1^{\underline{st}}$ inductive step

Let $a \in \mathcal{A}$ which is maximal. Then $f_a \in G$. If $D_a = \emptyset$ we define $g_a^1 = 0$ and $g_a^2 = 0$. If $D_a \neq \emptyset$ we set

$$E_a = \{k \in D_a : \; |f_a(x_k)| > \varepsilon\} \;\text{ and }\; F_a = D_a \setminus E_a.$$

From our assumption we have that $\#(E_a) \leq n_{j_0 - 1}$ and we define

$$g_a^1 = \sum_{k \in E_a} e_k^* \;\text{ and }\; g_a^2 = \sum_{k \in F_a} \varepsilon e_k^*.$$

We observe that $g_a^1 \in C_{j_0}$ and $\|g_a^2\|_\infty \leq \varepsilon$. Inequality (vii) is easily checked (see Proposition 4.3 of [AT1]).

General inductive step

Let $a \in \mathcal{A}$ and suppose that g_γ^1 and g_γ^2 have been defined for every $\gamma > a$ satisfying the inductive assumptions. If $D_a = \emptyset$ we set $g_a^1 = 0$ and $g_a^2 = 0$. In the remainder of the proof we assume that $D_a \neq \emptyset$. We consider the following three cases:

$1 \overset{st}{=}$ case The functional f_a is of type II.

Let $f_a = \sum_{\beta \in S_a} r_\beta f_\beta$ where $r_\beta \in \mathbb{Q}^+$ are such that $\sum_{\beta \in S_a} r_\beta = 1$. In this case, we have that $D_a = \bigcup_{\beta \in S_a} D_\beta$. We define

$$g_a^1 = \sum_{\beta \in S_a} r_\beta g_\beta^1 \ \text{ and } \ g_a^2 = \sum_{\beta \in S_a} r_\beta g_\beta^2.$$

For the proof of inequality (vii) see Proposition 4.3 of [AT1].

$2 \overset{nd}{=}$ case The functional f_a is of type I with $w(f) = m_{j_0}$.

In this case D_a is an interval of the natural numbers (Lemma A.9(ii)). Let $k_a \in D_a$ be such that $|\lambda_{k_a}| = \max_{k \in D_a} |\lambda_k|$. We define

$$g_a^1 = e_{k_a}^* \ \text{ and } \ g_a^2 = \sum_{k \in D_a} \varepsilon e_k^*.$$

Inequality (vii) is easily established.

$3 \overset{rd}{=}$ case The functional f_a is of type I with $w(f) = m_j$ for $j \neq j_0$.

Then $f_a = \frac{1}{m_j} \sum_{\beta \in S_a} f_\beta$ and the family $\{f_\beta :\ \beta \in S_a\}$ is a family of successive functionals with $\#(S_a) \leq n_j$. We set

$$\begin{aligned} E_a &= \{k :\ a \in \mathcal{A}_k \text{ and } \operatorname{supp} f_a \cap \operatorname{ran} x_k \neq \emptyset\} \\ &\quad (= \{k \in D_a \setminus \bigcup_{\beta \in S_a} D_\beta :\ \operatorname{supp} f_a \cap \operatorname{ran} x_k \neq \emptyset\}). \end{aligned}$$

We consider the following partition of E_a.

$$E_a^2 = \{k \in E_a :\ m_{j_{k+1}} \leq m_j\} \ \text{ and } \ E_a^1 = E_a \setminus E_a^2.$$

We define

$$g_a^2 = \sum_{k \in E_a^2} \varepsilon e_k^* + \sum_{\beta \in S_a} g_\beta^2.$$

Observe that $\|g_a^2\|_\infty \leq \varepsilon$. Let $E_a^1 = \{k_1 < k_2 < \cdots < k_l\}$. From the definition of E_a^1 we get that $m_j < m_{j_{k_2}} < \cdots < m_{j_{k_l}}$. We set

$$k_a = k_1 \ \text{ and } \ g_a^1 = e_{k_a}^* + h_a \ \text{ where } \ h_a = \frac{1}{m_j}(\sum_{i=2}^{l} e_{k_i}^* + \sum_{\beta \in S_a} g_\beta^1)$$

(The term $e_{k_a}^*$ does not appear if $E_a^1 = \emptyset$.)

For the verification of inequality (vii) see Proposition 4.3 of [AT1].

It remains to show that $h_a \in \operatorname{conv}_{\mathbb{Q}}\{h \in D'_{j_0} :\ w(h) = m_j\}$ By the second part of Lemma A.9(iii), for every $k \in E_a$ there exists an element of the set $N = \{\min \operatorname{supp} f_\beta, \max \operatorname{supp} f_\beta :\ \beta \in S_a\}$ belonging to $\operatorname{ran} x_k$. Hence $\#(E_a^1) \leq \#(E_a) \leq 2n_j$.

We next show that $h_a \in \mathrm{conv}_{\mathbb{Q}}\{g \in D'_{j_0} :\ w(g) = m_j\}$. We first examine the case that for every $\beta \in S_a$ the functional f_β is not of type II. Then for every $\beta \in S_a$ one of the following holds:

(i) $f_\beta \in G$. In this case $g^1_\beta \in C_{j_0}$ (by the first inductive step).

(ii) f_β is of type I with $w(f_\beta) = m_{j_0}$. In this case $g^1_\beta = e^*_{k_\beta} \in D'_{j_0}$.

(iii) f_β is of type I with $w(f_\beta) = m_j$ for $j \neq j_0$. In this case $g^1_\beta = e^*_{k_\beta} + h_\beta$ (or $g^1_\beta = h_\beta$) where $h_\beta \in \mathrm{conv}_{\mathbb{Q}}(D'_{j_0})$ and $k_\beta \notin \mathrm{supp}\, h_\beta$. We set $E^1_\beta = \{n \in \mathbb{N} :\ n < k_\beta\}$, $E^2_\beta = \{n \in \mathbb{N} :\ n > k_\beta\}$ and $h^1_\beta = E^1_\beta h_\beta$, $h^2_\beta = E^2_\beta h_\beta$. The functionals h^1_β, $e^*_{t_\beta}$, h^2_β are successive and belong to $D_{j_0} = \mathrm{conv}_{\mathbb{Q}}(D'_{j_0})$.

We set

$$\begin{aligned} T^1_a &= \{\beta \in S_a :\ f_\beta \in G\} \\ T^2_a &= \{\beta \in S_a :\ f_\beta \text{ of type } I \text{ and } w(f_\beta) = m_{j_0}\} \\ T^3_a &= \{\beta \in S_a :\ f_\beta \text{ of type } I \text{ and } w(f_\beta) \neq m_{j_0}\}. \end{aligned}$$

The family of successive (see Lemma A.9(iii)) functionals of D_{j_0},

$$\begin{aligned} &\{e^*_{k_i} :\ i = 2, \ldots, l\} \cup \{g^1_\beta :\ \beta \in T^1_a\} \cup \{g^1_\beta :\ \beta \in T^2_a\} \cup \\ &\quad \cup \{h^1_\beta :\ \beta \in T^3_a\} \cup \{e^*_{k_\beta} :\ \beta \in T^3_a\} \cup \{h^2_\beta :\ \beta \in T^3_a\} \end{aligned}$$

has cardinality $\leq 5n_j$, thus we get that $h_a \in D_{j_0}$ with $w(h_a) = m_j$. Therefore from Remark A.3 we get that

$$h_a \in \mathrm{conv}_{\mathbb{Q}}\{h \in D'_{j_0} :\ w(h) = m_j\}.$$

For the case that for some $\beta \in S_a$ the functional f_β is of type II see [AT1] Proposition 4.3. $\square$

Proof of Proposition 1.7. The proof is an application of the basic inequality (Proposition A.5) and Lemma A.4. Indeed, let $f \in D$ with $w(f) = m_i$. Proposition A.5 yields the existence of a functional h_1 with $h_1 \in \mathrm{conv}_{\mathbb{Q}}\{h \in D'_{j_0} :\ w(h) = m_i\}$, a $t \in \mathbb{N}$ and a $h_2 \in c_{00}(\mathbb{N})$ with $\|h_2\|_\infty \leq \varepsilon$, such that

$$|f(\frac{1}{n_{j_0}} \sum_{k=1}^{n_{j_0}} x_k)| \leq C(e^*_t + h_1 + h_2)\big(\frac{1}{n_{j_0}} \sum_{k=1}^{n_{j_0}} e_k\big).$$

If $i \geq j_0$ we get that $|f(\frac{1}{n_{j_0}} \sum_{k=1}^{n_{j_0}} x_k)| \leq C(\frac{1}{n_{j_0}} + \frac{1}{m_i} + \varepsilon) < \frac{C}{n_{j_0}} + \frac{C}{m_i} + C\varepsilon$. If $i < j_0$, using Lemma A.4 we get that $|f(\frac{1}{n_{j_0}} \sum_{k=1}^{n_{j_0}} x_k)| \leq C(\frac{1}{n_{j_0}} + \frac{2}{m_i \cdot m_{j_0}} + \varepsilon) < \frac{3C}{m_i \cdot m_{j_0}}$.

In order to prove 2) let $(b_k)_{k=1}^{n_{j_0}}$ be scalars with $|b_k| \leq 1$ such that (2) is satisfied. Then condition (22) of the basic inequality is satisfied for the linear combination $\frac{1}{n_{j_0}} \sum_{k=1}^{n_{j_0}} b_k x_k$ and thus for every $f \in D$ with $w(f) = m_i$, $i \neq j_0$, there exist a $t \in \mathbb{N}$ and $h_1, h_2 \in c_{00}(\mathbb{N})$ with h_1, h_2 having nonnegative coordinates and $\|h_2\|_\infty \le \varepsilon$ such that

$$\begin{aligned} |f(\frac{1}{n_{j_0}}\sum_{k=1}^{n_{j_0}} b_k x_k)| &\le C(e_t^* + h_1 + h_2)\big(\frac{1}{n_{j_0}}\sum_{k=1}^{n_{j_0}} |b_k| e_k\big) \\ &\le C(e_t^* + h_1 + h_2)\big(\frac{1}{n_{j_0}}\sum_{k=1}^{n_{j_0}} e_k\big) \end{aligned}$$

with h_1 being a rational convex combination $h_1 = \sum r_l \tilde{h_l}$ and for each l the functional $\tilde{h_l}$ belongs to D'_{j_0} with $w(\tilde{h_l}) = m_i$ and has a tree $T_{\tilde{h_l}} = (f_a^l)_{a\in\mathcal{C}_l}$ with $w(f_a^l) \neq m_{j_0}$ for all $a \in \mathcal{C}_l$. Using the second part of Lemma A.4 we deduce that

$$|f(\frac{1}{n_{j_0}}\sum_{k=1}^{n_{j_0}} b_k x_k)| \le C(\frac{1}{n_{j_0}} + \frac{1}{m_{j_0}^2} + \varepsilon) < \frac{4C}{m_{j_0}^2}.$$

For $f \in D$ with $w(f) = m_{j_0}$ it follows from condition (2) that $|f(\frac{1}{n_{j_0}}\sum_{k=1}^{n_{j_0}} b_k x_k)| \le \frac{C}{n_{j_0}}(1 + \frac{2}{m_{j_0}^2} n_{j_0}) < \frac{4C}{m_{j_0}^2}$. $\square$

Appendix B. The James tree spaces $JT_{\mathcal{F}_{2,s}}$, $JT_{\mathcal{F}_2}$ and $JT_{\mathcal{F}_s}$

In this part we continue the study of the James Tree spaces initialized in Section 3. We give a slightly different definition of JTG sequences and then we define the space $JT_{\mathcal{F}_2}$ exactly as in section 3. We also define the spaces $JT_{\mathcal{F}_s}$, $JT_{\mathcal{F}_{2,s}}$. We prove that $JT_{\mathcal{F}_2}$ is ℓ_2 saturated while $JT_{\mathcal{F}_s}$, $JT_{\mathcal{F}_{2,s}}$ are c_0 saturated. We also give an example of $JT_{\mathcal{F}_2}$, defined for a precise family $(F_j)_j$ such that the basis $(e_n)_{n\in\mathbb{N}}$ of the space is normalized weakly null and for every subsequence $(e_n)_{n\in M}$, $M \in [\mathbb{N}]$ the space $X_M = \overline{\text{span}}\{e_n :\ n \in M\}$ has nonseparable dual. As we have mentioned before the study of the James Tree spaces does not require techniques related to HI constructions.

Definition B.1. (JTG families) A sequence $(F_j)_{j=0}^\infty$ of subsets of $c_{00}(\mathbb{N})$ is said to be a *James Tree Generating family* (JTG family) provided that it satisfies the following conditions:

(A) $F_0 = \{\pm e_n^* :\ n \in \mathbb{N}\}$ and each F_j is nonempty, countable, symmetric, compact in the topology of pointwise convergence and closed under restrictions to intervals of $\mathbb{N}$.

(B) Setting $\tau_j = \sup\{\|f\|_\infty :\ f \in F_j\}$, the sequence $(\tau_j)_{j\in\mathbb{N}}$ is strictly decreasing and $\sum\limits_{j=1}^{\infty} \tau_j \le 1$.

(C) For every block sequence $(x_k)_{k\in\mathbb{N}}$ of $c_{00}(\mathbb{N})$, every $j = 0, 1, 2, \ldots$ and every $\delta > 0$ there exists a vector $x \in \text{span}\{x_k :\ k \in \mathbb{N}\}$ such that

$$\delta \cdot \sup\{f(x) :\ f \in \bigcup_{i=0}^{\infty} F_i\} > \sup\{f(x) :\ f \in F_j\}.$$

We set $F = \bigcup\limits_{j=0}^{\infty} F_j$. The set F defines a norm $\|\ \|_F$ on $c_{00}(\mathbb{N})$ by the rule

$$\|x\|_F = \sup\{f(x) :\ f \in F\}.$$

The space Y_F is the completion of the space $(c_{00}(\mathbb{N}), \| \|_F)$.

Examples B.2. We provide some examples of JTG families.

(i) The first example is what we call the Maurey-Rosenthal JTG family, related to the first construction of a normalized weakly null sequence with no unconditional subsequence ([MR]). In particular the norming set for their example is the set $F = \bigcup_{j=0}^{\infty} F_j$ together with the $\sigma_{\mathcal{F}}$ special functionals resulting from the family F. We proceed defining the sets $(F_j)_{j=0}^{\infty}$.

Let $(k_j)_{j\in\mathbb{N}}$ be a strictly increasing sequence of integers such that

$$\sum_{j=1}^{\infty}\sum_{n\neq j}\min\big\{\frac{\sqrt{k_n}}{\sqrt{k_j}}, \frac{\sqrt{k_j}}{\sqrt{k_n}}\big\} \leq 1.$$

We set $F_0 = \{\pm e_n^* :\ n \in \mathbb{N}\}$ while for $j = 1, 2, \dots$ we set

$$F_j = \big\{\frac{1}{\sqrt{k_j}}\big(\sum_{i\in F} \pm e_i^*\big) :\ \emptyset \neq F \subset \mathbb{N},\ \#(F) \leq k_j\big\} \cup \big\{0\big\}.$$

The above conditions (1) and (2) for the sequence $(k_j)_{j\in\mathbb{N}}$ easily yield that $(F_j)_{j=0}^{\infty}$ is a JTG family.

(ii) The second example is the family introduced in Section 4. For completeness we recall its definition. Let $(m_j)_{j\in\mathbb{N}}$ and $(n_j)_{j\in\mathbb{N}}$ defined as follows:

- $m_1 = 2$ and $m_{j+1} = m_j^5$.
- $n_1 = 4$, and $n_{j+1} = (5n_j)^{s_j}$ where $s_j = \log_2 m_{j+1}^3$.

We set $F_0 = \{\pm e_n^* :\ n \in \mathbb{N}\}$ and for $j = 1, 2, \dots$ we set

$$F_j = \big\{\frac{1}{m_{2j-1}^2}\sum_{i\in I} \pm e_i^* :\ \#(I) \leq \frac{n_{2j-1}}{2}\big\} \cup \big\{0\big\}.$$

We shall show that the sequence $(F_j)_{j=0}^{\infty}$ is a JTG family. Conditions (A), (B) of Definition B.1 are obviously satisfied. Suppose that condition (C) fails. Then for some $j \in \mathbb{N}$, there exists a block sequence $(x_k)_{k\in\mathbb{N}}$ in $c_{00}(\mathbb{N})$ with $\|x_k\|_F = 1$ and a $\delta > 0$ such that $\delta\|\sum a_k x_k\|_F \leq \|\sum a_k x_k\|_{F_j}$ for every sequence of scalars $(a_k)_{k\in\mathbb{N}} \in c_{00}(\mathbb{N})$. We observe that $\|x_k\|_\infty \geq \frac{2\delta m_{2j-1}^2}{n_{2j-1}}$ for all k. Indeed, if $\|x_k\|_\infty < \frac{2\delta m_{2j-1}^2}{n_{2j-1}}$ then for every $f \in F_j$, $f = \frac{1}{m_{2j-1}^2}\sum_{i\in I} \pm e_i^*$, with $\#(I) \leq \frac{n_{2j-1}}{2}$ we would have that $|f(x_k)| \leq \frac{1}{m_{2j-1}^2}\sum_{i\in I}|e_i^*(x_k)| < \frac{1}{m_{2j-1}^2}\frac{n_{2j-1}}{2}\frac{2\delta m_{2j-1}^2}{n_{2j-1}} = \delta$ which yields that $\|x_k\|_{F_j} < \delta$, a contradiction. Hence for each k we may select a $t_k \in \operatorname{supp} x_k$ such that $|e_{t_k}^*(x_k)| \geq \frac{2\delta m_{2j-1}^2}{n_{2j-1}}$. Since the sequence $(\frac{n_{2i-1}}{m_{2i-1}^2})_{i\in\mathbb{N}}$ increases to infinity we may choose a $j' \in \mathbb{N}$ such that $\delta^2 \frac{n_{2j'-1}}{m^2_{2j'-1}} > (\frac{n_{2j-1}}{m^2_{2j-1}})^2$. We consider the vector $y = \sum_{k=1}^{n_{2j'-1}/2} x_k$. We have that $\delta \|y\|_F \geq \delta \frac{1}{m^2_{2j'-1}} \sum_{k=1}^{n_{2j'-1}/2} |e^*_{t_k}(x_k)| \geq \delta \frac{1}{m^2_{2j'-1}} \frac{n_{2j'-1}}{2} \frac{2\delta m^2_{2j-1}}{n_{2j-1}} > \frac{1}{m^2_{2j-1}} n_{2j-1}$. On the other hand $\|y\|_{F_j} \leq \frac{1}{m^2_{2j-1}} \frac{n_{2j-1}}{2}$, a contradiction.

(iii) Another example of a JTG family has been given in Definition 6.4 and we have used it to define the ground set for the space $\mathfrak{X}_{\mathcal{F}_s}$.

Remarks B.3. (i) The standard basis $(e_n)_{n\in\mathbb{N}}$ of $c_{00}(\mathbb{N})$ is a normalized bimonotone Schauder basis of the space Y_F.

(ii) The set F is compact in the topology of pointwise convergence. Indeed, let $(f_n)_{n\in\mathbb{N}}$ be a sequence in F. There are two cases. Either there exists $j_0 \in \mathbb{N}$ such that the set F_{j_0} contains a subsequence of $(f_n)_{n\in\mathbb{N}}$ in which case the compactness of F_{j_0} yields the existence of a further subsequence converging pointwise to some $f \in F_{j_0}$, otherwise if no such j_0 exists, then we may find a subsequence $(f_{k_n})_{n\in\mathbb{N}}$ and a strictly increasing sequence $(i_n)_{n\in\mathbb{N}}$ of integers with $f_{k_n} \in F_{i_n}$ and thus $\|f_{k_n}\|_\infty \leq \tau_{i_n}$ for all n. Since condition (B) of Definition B.1 yields that $\tau_n \to 0$ we get that $f_{k_n} \xrightarrow{p} 0 \in F$.

(iii) The fact that F is countable and compact yields that the space $(C(F), \| \ \|_\infty)$ is c_0 saturated [BP]. It follows that the space Y_F is also c_0 saturated, since Y_F is isometric to a subspace of $(C(F), \| \ \|_\infty)$.

(iv) For each j we consider the seminorm $\| \ \|_{F_j} : c_{00}(\mathbb{N}) \to \mathbb{R}$ defined by $\|x\|_{F_j} = \sup\{|f(x)| : \ f \in F_j\}$. In general $\| \ \|_{F_j}$ is not a norm. Defining Y_{F_j} to be the completion of the space $(c_{00}(\mathbb{N}) \ , \| \ \|_{F_j})$, condition (C) of Definition B.1 is equivalent to saying that the identity operator $I : Y_F \to Y_{F_j}$ is strictly singular.

Furthermore, observe that setting $H_j = \cup_{i=0}^{j} F_i$ the identity operator $I : Y_F \to Y_{H_j}$ (Y_{H_j} is similarly defined) is also strictly singular. Indeed, let $(x_k)_{k\in\mathbb{N}}$ be a block sequence of $c_{00}(\mathbb{N})$ and let $\delta > 0$. We choose a block sequence $(x^0_k)_{k\in\mathbb{N}}$ of $(x_k)_{k\in\mathbb{N}}$ with $\|x^0_k\|_F = 1$ and $\sum_{k=1}^{\infty} \|x^0_k\|_{F_1} < \delta$. Then for every $x \in \operatorname{span}\{x^0_k : \ k \in \mathbb{N}\}$ we have that $\delta\|x\|_F \geq \|x\|_{F_0}$. We then select a block sequence $(x^1_k)_{k\in\mathbb{N}}$ of $(x^0_k)_{k\in\mathbb{N}}$ such that $\delta\|x\|_F \geq \|x\|_{F_1}$ for every $x \in \operatorname{span}\{x^1_k : \ k \in \mathbb{N}\}$. Following this procedure, after $j+1$ steps we may select a block sequence $(x^j_k)_{k\in\mathbb{N}}$ of $(x_k)_{k\in\mathbb{N}}$ such that $\delta\|x\|_F \geq \|x\|_{F_i}$ for $i = 1, \ldots, j$ and thus $\delta\|x\|_F \geq \|x\|_{H_j}$ for every $x \in \operatorname{span}\{x^j_k : \ k \in \mathbb{N}\}$.

Next using the $\sigma_{\mathcal{F}}$ coding defined in Definition 3.2 we introduce the $\sigma_{\mathcal{F}}$ special sequences and functionals in the same manner as in Definition 3.3. For a $\sigma_{\mathcal{F}}$ special functional x^* the index $\operatorname{ind}(x^*)$ has the analogous meaning. Finally we denote by $\mathcal{S}$ the set of all finitely supported $\sigma_{\mathcal{F}}$ special functionals.

The next proposition is an immediate consequence of the above definition and describes the tree-like interference of two $\sigma_{\mathcal{F}}$ special sequences.

Proposition B.4. Let $(f_i)_i$, $(h_i)_i$ be two distinct $\sigma_{\mathcal{F}}$ special sequences. Then $\mathrm{ind}(f_i) \neq \mathrm{ind}(h_j)$ for $i \neq j$ while there exists i_0 such that $f_i = h_i$ for all $i < i_0$ and $\mathrm{ind}(f_i) \neq \mathrm{ind}(h_i)$ for $i > i_0$.

Definition B.5. (The norming sets $\mathcal{F}_{2,s}$, $\mathcal{F}_2$, $\mathcal{F}_s$) Let $(F_j)_{j=0}^{\infty}$ be a JTG family. We set

$$\begin{aligned}\mathcal{F}_2 \quad = \quad & F_0 \cup \Big\{ \sum_{k=1}^{d} a_k x_k^* : \ a_k \in \mathbb{Q}, \ \sum_{k=1}^{d} a_k^2 \leq 1, \ x_k^* \in \mathcal{S} \cup \bigcup_{i=1}^{\infty} F_i, \ k = 1, \dots, d \\ & \quad \text{with } (\mathrm{ind}(x_k^*))_{k=1}^{d} \text{ pairwise disjoint} \Big\}\end{aligned}$$

$$\begin{aligned}\mathcal{F}_{2,s} \qquad & F_0 \cup \Big\{ \sum_{k=1}^{d} a_k x_k^* : \ a_k \in \mathbb{Q}, \ \sum_{k=1}^{d} a_k^2 \leq 1, \ x_k^* \in \mathcal{S} \cup \bigcup_{i=1}^{\infty} F_i, \ k = 1, \dots, d \\ & \quad \text{with } (\mathrm{ind}(x_k^*))_{k=1}^{d} \text{ pairwise disjoint and } \min\operatorname{supp} x_k^* \geq d \Big\},\end{aligned}$$

and

$$\begin{aligned}\mathcal{F}_s \quad = \quad & F_0 \cup \{ \sum_{k=1}^{d} \varepsilon_k x_k^* : \ \varepsilon_1, \dots, \varepsilon_d \in \{-1, 1\}, \ x_k^* \in \mathcal{S} \cup \bigcup_{i=1}^{\infty} F_i, \ k = 1, \dots, d \\ & \quad \text{with } (\mathrm{ind}(x_i^*))_{i=1}^{d} \text{ pairwise disjoint and } \min\operatorname{supp} x_i^* \geq d, \ d \in \mathbb{N} \}.\end{aligned}$$

The space $JT_{\mathcal{F}_{2,s}}$ is defined as the completion of the space $(c_{00}(\mathbb{N}) \ , \| \ \|_{\mathcal{F}_{2,s}})$, the space $JT_{\mathcal{F}_2}$ is defined to be the completion of the space $(c_{00}(\mathbb{N}) \ , \| \ \|_{\mathcal{F}_2})$ while $JT_{\mathcal{F}_s}$ the completion of $(c_{00}(\mathbb{N}) \ , \| \ \|_{\mathcal{F}_s})$ (where $\|x\|_{\mathcal{F}_*} = \sup\{f(x) : \ f \in \mathcal{F}\}$ for $x \in c_{00}(\mathbb{N})$, for either $\mathcal{F}_* = \mathcal{F}_2$ or $\mathcal{F}_* = \mathcal{F}_{2,s}$ or $\mathcal{F}_* = \mathcal{F}_s$). For a functional $f \in \mathcal{F}_* \setminus F_0$ of the form $f = \sum\limits_{k=1}^{l} a_k x_k^*$ the set of its indices $\mathrm{ind}(f)$ is defined to be the set $\mathrm{ind}(f) = \bigcup\limits_{k=1}^{l} \mathrm{ind}(x_k^*)$.

Remark B.6. The standard basis $(e_n)_{n\in\mathbb{N}}$ of $c_{00}(\mathbb{N})$ is a normalized bimonotone Schauder basis for the space $JT_{\mathcal{F}_*}$.

Let's observe that the only difference between the definition of $\mathcal{F}_{2,s}$ and that of $\mathcal{F}_2$ is the way we connect the $\sigma_{\mathcal{F}}$ special functionals. In the case of $\mathcal{F}_2$ the $\sigma_{\mathcal{F}}$ special functionals are connected more freely than in $\mathcal{F}_{2,s}$ and obviously $\mathcal{F}_{2,s} \subset \mathcal{F}_2$. This difference leads the spaces $JT_{\mathcal{F}_{2,s}}$ and $JT_{\mathcal{F}_2}$ to have extremely different structures. We study the structure of these two spaces as well as the structure of $JT_{\mathcal{F}_s}$. Namely we have the following theorem.

Theorem B.7.
(i) The space $JT_{\mathcal{F}_{2,s}}$ is c_0 saturated.
(ii) The space $JT_{\mathcal{F}_2}$ is ℓ_2 saturated.
(iii) The space $JT_{\mathcal{F}_s}$ is c_0 saturated.

Proposition B.4 yields that the set of all finite $\sigma_{\mathcal{F}}$ special sequences is naturally endowed with a tree structure. The set of infinite branches of this tree structure is identified with the set of all infinite $\sigma_{\mathcal{F}}$ special sequences. For such a branch $b = (f_1, f_2, \dots)$ the functional $b^* = \lim\limits_{d} \sum\limits_{i=1}^{d} f_i$ (where the limit is taken in the pointwise topology) is a cluster point of the sets $\mathcal{F}_2$, $\mathcal{F}_{2,s}$

and $\mathcal{F}_s$ and hence belongs to the unit balls of the dual spaces $JT^*_{\mathcal{F}_2}$ $JT^*_{\mathcal{F}_{2,s}}$ and $JT^*_{\mathcal{F}_s}$. Let also point out that a $\sigma_{\mathcal{F}}$ special functional x^* is either finite or takes the form Eb^* for some branch b and some interval E. Furthermore, it is easy to check that the set $\{Ex^* :\ E \text{ interval}, x^*\ \sigma_{\mathcal{F}} \text{ special functional}\}$ is closed in the pointwise topology.

Our main goal in this section is to prove Theorem B.7. Many of the Lemmas used in proving this theorem are common for $JT_{\mathcal{F}_{2,s}}$, $JT_{\mathcal{F}_2}$ and $JT_{\mathcal{F}_s}$. For this reason it is convenient to use the symbol $\mathcal{F}_*$ when stating or proving a property which is valid for $\mathcal{F}_* = \mathcal{F}_{2,s}$, $\mathcal{F}_* = \mathcal{F}_2$ and $\mathcal{F}_* = \mathcal{F}_s$.

Lemma B.8. The identity operator $I : JT_{\mathcal{F}_*} \to Y_F$ is strictly singular.

Proof. Assume the contrary. Then there exists a block subspace Y of $JT_{\mathcal{F}_*}$ such that the identity operator $I : (Y, \|\ \|_{\mathcal{F}_*}) \to (Y, \|\ \|_F)$ is an isomorphism. Since Y_F is c_0 saturated (Remark B.3 (iii)) we may assume that $(Y, \|\ \|_{\mathcal{F}_*})$ is is spanned by a block basis which is equivalent to the standard basis of c_0. Using property (C) of Definition B.1 and Remark B.3 (iv) we inductively choose a normalized block sequence $(x_n)_{n\in\mathbb{N}}$ in $(Y, \|\ \|_{\mathcal{F}_*})$ and a strictly increasing sequence $(j_n)_{n\in\mathbb{N}}$ of integers such that for some δ determined by the isomorphism, the following hold:

(i) $\|x_n\|_{F_{j_n}} > \delta$.
(ii) $\|x_{n+1}\|_{\bigcup_{k=1}^{j_n} F_k} < \delta$.

From (i) and (ii) and the definition of each $\mathcal{F}_*$ we easily get that $\|x_1 + \cdots + x_n\| \xrightarrow{n} \infty$. This is a contradiction since $(x_n)_{n\in\mathbb{N}}$, being a normalized block basis of a sequence equivalent to the standard basis of c_0, is also equivalent to the standard basis of c_0. $\square$

The following lemma, although it refers exclusively to the functional b^*, its proof is crucially depended on the fact that in $\mathcal{F}_*$ we connect the special functionals under certain norms. A similar result is also obtained in [AT1] (Lemma 10.6).

Lemma B.9. Let $(x_n)_{n\in\mathbb{N}}$ be a bounded block sequence in $JT_{\mathcal{F}_*}$. Then there exists an $L \in [\mathbb{N}]$ such that for every branch b the limit $\lim\limits_{n\in L} b^*(x_n)$ exists. In particular, if the sequence $(x_n)_{n\in\mathbb{N}}$ is seminormalized (i.e. $\inf \|x_n\|_{\mathcal{F}_*} > 0$) and $L = \{l_1 < l_2 < l_3 < \cdots\}$ then the sequence $y_n = \frac{x_{l_{2n-1}} - x_{l_{2n}}}{\|x_{l_{2n-1}} - x_{l_{2n}}\|}$ satisfies $\|y_n\|_{\mathcal{F}_*} = 1$ and $\lim\limits_n b^*(y_n) = 0$ for every branch b.

Proof. We first prove the following claim.

Claim. For every $\varepsilon > 0$ and $M \in [\mathbb{N}]$ there exists $L \in [M]$ and a finite collection of branches $\{b_1, \ldots, b_l\}$ such that for every branch b with $b \notin \{b_1, \ldots, b_l\}$ we have that $\limsup\limits_{n\in L} |b^*(x_n)| \le \varepsilon$.

Proof of the claim. Assume the contrary. Then we may inductively construct a sequence $M_1 \supset M_2 \supset M_3 \cdots$ of infinite subsets of $\mathbb{N}$ and a sequence $b_1, b_2, b_3, \ldots$ of pairwise different branches satisfying $|b_i^*(x_n)| > \varepsilon$ for all $n \in M_i$.

We set $C = \sup_n \|x_n\|_{\mathcal{F}_*}$ and we consider $k > \frac{C}{\varepsilon}$. Since the branches $b_1, b_2, \ldots, b_{k^2}$ are pairwise different we may choose an infinite interval E with $\min E \geq k^2$ such that the functionals $(Eb_i^*)_{i=1}^{k^2}$ have disjoint indices. We also consider any $n \in M_{k^2}$ with $\operatorname{supp} x_n \subset E$ and we set $\varepsilon_i = \operatorname{sgn} b_i^*(x_n)$ for $i = 1, 2, \ldots, k^2$. Then the functional $f = \sum_{i=1}^{k^2} \frac{\varepsilon_i}{k} b_i^*$ belongs to $B_{JT^*_{\mathcal{F}_*}}$. Therefore

$$\|x_n\|_{\mathcal{F}_*} \geq f(x_n) = \sum_{i=1}^{k^2} \frac{1}{k} |b_i^*(x_n)| \geq \sum_{i=1}^{k^2} \frac{1}{k} \cdot \varepsilon = k \cdot \varepsilon > C,$$

a contradiction completing the proof of the claim. □

Using the claim we inductively select a sequence $L_1 \supset L_2 \supset L_3 \supset \cdots$ of infinite subsets of $\mathbb{N}$ and a sequence $B_1, B_2, B_3, \ldots$ of finite collections of branches such that for every branch $b \notin B_i$ we have that $|b^*(x_n)| < \frac{1}{i}$ for all $n \in L_i$. We then choose a diagonal set L_0 of the nested sequence $(L_i)_{i \in \mathbb{N}}$. Then for every branch b not belonging to $B = \bigcup_{i=1}^{\infty} B_i$ we have that $\lim_{n \in L_0} b^*(x_n) = 0$. Since the set B is countable, we may choose, using a diagonalization argument, an $L \in [L_0]$ such that the sequence $(b^*(x_n))_{n \in L}$ converges for every $b \in F$. The set L clearly satisfies the conclusion of the lemma. □

Combining Lemma B.8 and Lemma B.9 we get the following.

Corollary B.10. Every block subspace of $JT_{\mathcal{F}_*}$ contains a block sequence $(y_n)_{n \in \mathbb{N}}$ such that $\|y_n\|_{\mathcal{F}_*} = 1$, $\|y_n\|_F \overset{n \to \infty}{\longrightarrow} 0$ and $b^*(y_n) \overset{n \to \infty}{\longrightarrow} 0$ for every branch b.

Lemma B.11. Let Y be a block subspace of $JT_{\mathcal{F}_*}$ and let $\varepsilon > 0$. Then there exists a finitely supported vector $y \in Y$ such that $\|y\|_{\mathcal{F}_*} = 1$ and $|x^*(y)| < \varepsilon$ for every $\sigma_{\mathcal{F}}$ special functional x^*.

Proof. Assume the contrary. Then there exists a block subspace Y of $JT_{\mathcal{F}_*}$ and an $\varepsilon > 0$ such that

$$\varepsilon \cdot \|y\|_{\mathcal{F}_*} \leq \sup\{|x^*(y)| : \ x^* \text{ is a } \sigma_{\mathcal{F}} \text{ special functional}\} \tag{23}$$

for every $y \in Y$. Let $q > \frac{8}{\varepsilon^2}$. From Corollary B.10 we may select a block sequence $(y_n)_{n \in \mathbb{N}}$ in Y such that $\|y_n\|_{\mathcal{F}_*} = 1$, $\|y_n\|_F \overset{n \to \infty}{\longrightarrow} 0$ and $b^*(y_n) \overset{n \to \infty}{\longrightarrow} 0$ for every branch b. Observe also that $(y_n)_{n \in \mathbb{N}}$ is a separated sequence (Definition 3.11) hence from Lemma 3.12 we may assume passing to a subsequence that for every $\sigma_{\mathcal{F}}$ special functional x^* we have that $|x^*(y_n)| \geq \frac{1}{q^2}$ for at most two y_n. (Although Lemma 3.12 is stated for $JT_{\mathcal{F}_2}$ with small modifications in the proof remains valid for either $\mathcal{F}_s$ or $\mathcal{F}_{2,s}$.)

We set $t_1 = 1$. From (23) there exists a $\sigma_{\mathcal{F}}$ special functional y_1^* with $\operatorname{ran} y_1^* \subset \operatorname{ran} y_{t_1}$ such that $|y_1^*(y_{t_1})| > \frac{\varepsilon}{2}$. Setting $d_1 = \max \operatorname{ind} y_1^*$ we select t_2 such that $\|y_{t_2}\|_F < \frac{\varepsilon}{4d_1}$. Let z_2^* be a $\sigma_{\mathcal{F}}$ special functional with $\operatorname{ran} z_2^* \subset \operatorname{ran} y_{t_2}$ such that $|z_2^*(y_{t_2})| > \frac{3\varepsilon}{4}$. We write $z_2^* = x_2^* + y_2^*$ with $\operatorname{ind} x_2^* \subset \{1, \ldots, d_1\}$ and $\operatorname{ind} y_2^* \subset \{d_1 + 1, \ldots\}$. We have that $|x_2^*(y_{t_2})| \leq d_1 \|y_{t_2}\|_F < \frac{\varepsilon}{4}$ and thus $|y_2^*(y_{t_2})| > \frac{\varepsilon}{2}$. Following this procedure we select a finite collection

$(t_n)_{n=1}^{q^2}$ of integers and a finite sequence of $\sigma_{\mathcal{F}}$ special functionals $(y_n^*)_{n=1}^{q^2}$ with $\operatorname{ran} y_n^* \subset \operatorname{ran} y_{t_n}$ and $|y_n^*(y_{t_n})| > \frac{\varepsilon}{2}$ such that the sets of indices $(\operatorname{ind}(y_n^*))_{n=1}^{q^2}$ are pairwise disjoint. We consider the vector $y = y_{t_1} + y_{t_2} + \ldots + y_{t_{q^2}}$.

We set $\varepsilon_i = \operatorname{sgn} y_n^*(y_{t_n})$ for $i = 1, \ldots, q^2$. The functional $f = \sum_{n=1}^{q^2} \frac{\varepsilon_n}{q} y_n^*$ belongs to $\mathcal{F}_{2,s}$ ($\subset \mathcal{F}_2$), while $qf \in \mathcal{F}_s$. Therefore

$$\|y\|_{\mathcal{F}_*} \geq f(y_1 + y_2 + \ldots + y_{q^2}) \geq \frac{1}{q} \sum_{n=1}^{q^2} |y_n^*(y_{t_n})| > q\frac{\varepsilon}{2} > \frac{4}{\varepsilon}. \tag{24}$$

It is enough to show that $\sup\{|x^*(y)| : x^* \text{ is a } \sigma_{\mathcal{F}} \text{ special functional}\} \leq 3$ so as to derive a contradiction with (23) and (24). Let x^* be a $\sigma_{\mathcal{F}}$ special functional. Then from our assumptions that $|x^*(y_n)| > \frac{1}{q^2}$ for at most two y_n,

$$|x^*(y)| \leq 2 + (q^2 - 2)\frac{1}{q^2} < 3.$$

The proof of the lemma is complete. □

Theorem B.12. Let Y be a subspace of either $JT_{\mathcal{F}_{2,s}}$ or of $JT_{\mathcal{F}_s}$. Then for every $\varepsilon > 0$, there exists a subspace of Y which $1+\varepsilon$ isomorphic to c_0.

Proof. Let Y be a block subspace of $JT_{\mathcal{F}_{2,s}}$ or of $JT_{\mathcal{F}_s}$ and let $\varepsilon > 0$. Using Lemma B.11 we may inductively select a normalized block sequence $(y_n)_{n\in\mathbb{N}}$ in Y such that, setting $d_n = \max\operatorname{supp} y_n$ for each n and $d_0 = 1$, $|x^*(y_n)| < \frac{\varepsilon}{2^n d_{n-1}}$ for every $\sigma_{\mathcal{F}}$ special functional x^*.

We claim that $(y_n)_{n\in\mathbb{N}}$ is $1+\varepsilon$ isomorphic to the standard basis of c_0. Indeed, let $(\beta_n)_{n=1}^N$ be a sequence of scalars. We shall show that $\max_{1\leq n\leq N} |\beta_n| \leq \|\sum_{n=1}^N \beta_n y_n\|_{\mathcal{F}_*} \leq (1+\varepsilon) \max_{1\leq n\leq N} |\beta_n|$ for either $\mathcal{F}_* = \mathcal{F}_{2,s}$ or $\mathcal{F}_* = \mathcal{F}_s$. We may assume that $\max_{1\leq n\leq N} |\beta_n| = 1$. The left inequality follows directly from the bimonotonicity of the Schauder basis $(e_n)_{n\in\mathbb{N}}$ of $JT_{\mathcal{F}_*}$.

To see the right inequality we consider an arbitrary $g \in \mathcal{F}_*$. Then there exist $d \in \mathbb{N}$, $(x_i^*)_{i=1}^d$ in $\mathcal{S} \cup (\bigcup_{i=1}^\infty F_i)$ with $(\operatorname{ind}(x_i^*))_{i=1}^d$ pairwise disjoint and $\min\operatorname{supp} x_i^* \geq d$, such that $g = \sum_{i=1}^d a_i x_i^*$ with $\sum_{i=1}^d a_i^2 \leq 1$ in the case $\mathcal{F}_* = \mathcal{F}_{2,s}$, while $a_i \in \{-1, 1\}$ in the case $\mathcal{F}_* = \mathcal{F}_s$. Let n_0 be the minimum integer n such that $d \leq d_n$. Since $\min\operatorname{supp} g \geq d > d_{n_0-1}$ we get that $g(y_n) = 0$ for $n < n_0$. In either case we get that

$$\begin{aligned} g(\sum_{n=1}^N \beta_n y_n) &\leq |g(y_{n_0})| + \sum_{n=n_0+1}^N |g(y_n)| \leq 1 + \sum_{n=n_0+1}^N \sum_{i=1}^d |x_i^*(y_n)| \\ &< 1 + \sum_{n=n_0+1}^N d\frac{\varepsilon}{2^n d_{n-1}} < 1 + \sum_{n=n_0+1}^N \frac{\varepsilon}{2^n} < 1 + \varepsilon. \end{aligned}$$

The proof of the theorem is complete. □

Lemma B.13. For every $x \in c_{00}(\mathbb{N})$ and every $\varepsilon > 0$ there exists $d \in \mathbb{N}$ (denoted $d = d(x, \varepsilon)$) such that for every $g \in \mathcal{F}_2 \setminus F_0$ with $\mathrm{ind}(g) \cap \{1, \ldots, d\} = \emptyset$ we have that $|g(x)| < \varepsilon$.

Proof. Let $C = \|x\|_{\ell_1}$ be the ℓ_1 norm of the vector x. We choose $d \in \mathbb{N}$ such that $\sum_{l=d+1}^{\infty} \tau_l^2 < (\frac{\varepsilon}{C})^2$. Now let $g = \sum_{i=1}^{k} a_i x_i^* \in \mathcal{F}_2$, such that $\mathrm{ind}(g) \cap \{1, \ldots, d\} = \emptyset$. Each x_i^* takes the form $x_i^* = \sum_{j=1}^{r_i} x_{i,j}^*$, where for each i either $r_i = 1$ and $x_{i,1} \in \bigcup_{i=1}^{\infty} F_i$ or $(x_{i,j})_{j=1}^{r_i}$ is a $\sigma_{\mathcal{F}}$ special sequence, and the indices $(\mathrm{ind}(x_{i,j}^*))_{i,j}$ are pairwise different elements of $\{d+1, d+2, \ldots\}$. We get that

$$\begin{aligned}
|g(x)| &\leq \sum_{i=1}^{k} |a_i| \cdot |x_i^*(x)| \leq \big(\sum_{i=1}^{k} |a_i|^2)\big)^{1/2} \big(\sum_{i=1}^{k} |x_i^*(x)|^2\big)^{1/2} \\
&\leq 1 \cdot \big(\sum_{i=1}^{k} \|x\|_{\ell_1}^2 \|x_i^*\|_\infty^2\big)^{1/2} \leq C\big(\sum_{l=d+1}^{\infty} \tau_l^2\big)^{1/2} < \varepsilon.
\end{aligned}$$

□

Theorem B.14. For every subspace Y of $JT_{\mathcal{F}_2}$ and every $\varepsilon > 0$ there exists a subspace of Y which is $1 + \varepsilon$ isomorphic to ℓ_2.

Proof. Let Y be a block subspace of $JT_{\mathcal{F}_2}$ and let $\varepsilon > 0$. We choose a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive reals satisfying $\sum_{n=1}^{\infty} \varepsilon_n < \frac{\varepsilon}{2}$. We shall produce a block sequence $(x_n)_{n\in\mathbb{N}}$ in Y and a strictly increasing sequence of integers $(d_n)_{n\in\mathbb{N}}$ such that

(i) $\|x_n\|_{\mathcal{F}_2} = 1$.
(ii) For every $\sigma_{\mathcal{F}}$ special functional x^* we have that $|x^*(x_n)| < \frac{\varepsilon_n}{3d_{n-1}}$.
(iii) If $g = \sum_{i=1}^{k} a_i y_i^* \in \mathcal{F}_2$ is such that $\mathrm{ind}(g) \cap \{1, 2, \ldots, d_n\} = \emptyset$, then $|g(x_n)| < \varepsilon_n$.

The construction is inductive. We choose an arbitrary finitely supported vector $x_1 \in Y$ with $\|x_1\|_{\mathcal{F}_2} = 1$ and we set $d_1 = d(x_1, \varepsilon_1)$ (see the notation in the statement of Lemma B.13). Then, using Lemma B.11 we select a vector $x_2 \in Y \cap c_{00}(\mathbb{N})$ with $x_1 < x_2$ such that $\|x_2\|_{\mathcal{F}_2} = 1$ and $|x^*(x_2)| < \frac{\varepsilon_2}{3d_1}$ for every $\sigma_{\mathcal{F}}$ special functional x^*. We set $d_2 = d(x_2, \varepsilon_2)$. It is clear how the inductive construction proceeds. We shall show that for every sequence of scalars $(\beta_n)_{n=1}^{N}$ we have that

$$(1 - \varepsilon)\big(\sum_{n=1}^{N} \beta_n^2\big)^{1/2} \leq \|\sum_{n=1}^{N} \beta_n x_n\|_{\mathcal{F}_2} \leq (1 + \varepsilon)\big(\sum_{n=1}^{N} \beta_n^2\big)^{1/2}. \tag{25}$$

We may assume that $\sum_{n=1}^{N} \beta_n^2 = 1$.

We first show the left hand inequality of (25). For each n we choose $g_n \in \mathcal{F}_2$, $g_n = \sum_{i=1}^{l_n} a_{n,i} g_{n,i}$, with $\operatorname{ran} g_n \subset \operatorname{ran} x_n$ and

$$g_n(x_n) > 1 - \frac{\varepsilon}{3}. \tag{26}$$

For each (n,i) we write the functional $g_{n,i}$ as the sum of three successive functionals, $g_{n,i} = x^*_{n,i} + y^*_{n,i} + z^*_{n,i}$ such that $\operatorname{ind}(x^*_{n,i}) \subset \{1, \ldots, d_{n-1}\}$, $\operatorname{ind}(y^*_{n,i}) \subset \{d_{n-1}+1, \ldots, d_n\}$ and $\operatorname{ind}(z^*_{n,i}) \subset \{d_n+1, \ldots\}$. From the choice of the vector x_n and the definition $x^*_{n,i}$ we get that

$$|\sum_{i=1}^{l_n} a_{n,i} x^*_{n,i}(x_n)| \leq \sum_{i=1}^{l_n} |x^*_{n,i}(x_n)| \leq d_{n-1} \cdot \|x_n\|_F < d_{n-1} \frac{\varepsilon_n}{3 d_{n-1}} < \frac{\varepsilon}{3}. \tag{27}$$

The definition of the number d_n yields also that

$$|\sum_{i=1}^{l_n} a_{n,i} z^*_{n,i}(x_n)| < \frac{\varepsilon}{3}. \tag{28}$$

From (26), (27) and (28) we get that $g'_n(x_n) > 1 - \varepsilon$ where the functional $g'_n = \sum_{i=1}^{l_n} a_{n,i} y^*_{n,i}$, belongs to $\mathcal{F}_2$, satisfies $\operatorname{ran}(g'_n) \subset \operatorname{ran}(g_n) \subset \operatorname{ran}(x_n)$ and $\operatorname{ind}(g'_n) \subset \{d_{n-1}+1, \ldots, d_n\}$. We consider the functional $g = \sum_{n=1}^{N} \beta_n g'_n = \sum_{n=1}^{N} \sum_{i=1}^{l_n} \beta_n a_{n,i} y^*_{n,i}$. Since $\sum_n \sum_i (\beta_n a_{n,i})^2 \leq 1$ and the sets $(\operatorname{ind}(y^*_{n,i}))_{n,i}$ are pairwise disjoint we get that $g \in \overline{\mathcal{F}_2}^p \subset B_{JT^*_{\mathcal{F}_2}}$. Therefore

$$\|\sum_{n=1}^{N} \beta_n x_n\|_{\mathcal{F}_2} \geq g(\sum_{n=1}^{N} \beta_n x_n) \sum_{n=1}^{N} \beta_n^2 g'_n(x_n) > 1 - \varepsilon.$$

We next show the right hand inequality of (25). Let $(\beta_n)_{n=1}^N$ be any sequence of scalars such that $\sum_{n=1}^{N} \beta_n^2 \leq 1$. We consider an arbitrary $f \in \mathcal{F}_2$, and we shall show that $f(\sum_{n=1}^{N} \beta_n x_n) \leq 1 + \varepsilon$. Let $f = \sum_{i=1}^{k} a_i x^*_i$, where $(x^*_i)_{i=1}^k$ belong to $\mathcal{S} \cup (\bigcup_{i=1}^{\infty} F_i)$ with pairwise disjoint sets of indices and $\sum_{i=1}^{k} a_i^2 \leq 1$.

We partition the set $\{1, 2, \ldots, k\}$ in the following manner. We set

$$A_1 = \big\{ i \in \{1, 2, \ldots, k\} : \operatorname{ind}(x^*_i) \cap \{1, 2, \ldots, d_1\} \neq \emptyset \big\}.$$

If $A_1, \ldots, A_{n-1}$ have been defined we set

$$A_n = \big\{ i \in \{1, 2, \ldots, k\} : \operatorname{ind}(x^*_i) \cap \{1, 2, \ldots, d_n\} \neq \emptyset \big\} \setminus \bigcup_{i=1}^{n-1} A_i.$$

Finally we set $A_{N+1} = \{1, 2, \ldots, k\} \setminus \bigcup_{i=1}^{N} A_i$.

The sets $(A_n)_{n=1}^{N+1}$ are pairwise disjoint and $\#\big(\bigcup_{i=1}^{n} A_i\big) \leq d_n$ for $n = 1, 2, \dots, N$. We set

$$f_{A_n} = \sum_{i \in A_n} a_i x_i^*, \qquad i = 1, 2, \dots, N+1.$$

It is clear that $\|f_{A_n}\|_{JT^*_{\mathcal{F}_2}} \leq \big(\sum_{i \in A_n} a_i^2\big)^{1/2}$ for each n.

Let $n \in \{1, 2, \dots, N\}$ be fixed. We have that

$$|f(\beta_n x_n)| \leq \sum_{l=1}^{n-1} |f_{A_l}(x_n)| + |f_{A_n}(\beta_n x_n)| + |\sum_{l=n+1}^{N+1} f_{A_l}(x_n)|. \tag{29}$$

From condition (ii) we get that

$$\sum_{l=1}^{n-1} |f_{A_l}(x_n)| \leq \sum_{i \in \cup_{l=1}^{n-1} A_l} |x_i^*(x_n)| < d_{n-1} \cdot \frac{\varepsilon_n}{d_{n-1}} = \varepsilon_n. \tag{30}$$

On the other hand
$\mathrm{ind}(\sum_{l=n+1}^{N+1} f_{A_l}) \cap \{1, \dots, d_n\} = \mathrm{ind}(\sum_{l=n+1}^{N+1} \sum_{i \in A_l} a_i x_i^*) \cap \{1, \dots, d_n\} = \emptyset$ and thus condition (iii) yields that

$$|\sum_{l=n+1}^{N+1} f_{A_l}(x_n)| < \varepsilon_n. \tag{31}$$

Inequalities (29),(30) and (31) yield that

$$|f(\beta_n x_n)| < |f_{A_n}(\beta_n x_n)| + 2\varepsilon_n.$$

Therefore

$$\begin{aligned}
|f\big(\sum_{n=1}^{N} \beta_n x_n\big)| &\leq \sum_{n=1}^{N} |f(\beta_n x_n)| \leq \sum_{n=1}^{N} \big(|f_{A_n}(\beta_n x_n)| + 2\varepsilon_n\big) \\
&\leq \sum_{n=1}^{N} |\beta_n||f_{A_n}(x_n)| + 2\sum_{n=1}^{N} \varepsilon_n \\
&< \big(\sum_{n=1}^{N} |\beta_n|^2\big)^{1/2} \big(\sum_{n=1}^{N} |f_{A_n}(x_n)|^2\big)^{1/2} + \varepsilon \\
&\leq 1 \cdot \big(\sum_{n=1}^{N} \sum_{i \in A_n} a_i^2\big)^{1/2} + \varepsilon \leq 1 + \varepsilon.
\end{aligned}$$

The proof of the theorem is complete. □

Proposition B.15. The dual space $JT^*_{\mathcal{F}_*}$ is equal to the closed linear span of the set containing $(e_n^*)_{n \in \mathbb{N}}$ and b^* for every branch b,

$$JT^*_{\mathcal{F}_*} = \overline{\mathrm{span}}(\{e_n^* :\ n \in \mathbb{N}\} \cup \{b^* :\ b \text{ is a } \sigma_{\mathcal{F}} \text{ branch}\}).$$

Moreover the Schauder basis $(e_n)_{n \in \mathbb{N}}$ of the space $JT_{\mathcal{F}_*}$ is weakly null.

Proof. Since the space $JT_{\mathcal{F}_*}$ is c_0 saturated for $\mathcal{F}_* = \mathcal{F}_{2,s}$ or $\mathcal{F}_* = \mathcal{F}_s$ (Theorem B.12) or ℓ_2 saturated (for $\mathcal{F}_* = \mathcal{F}_2$) it contains no isomorphic copy of ℓ_1. Haydon's theorem yields that the unit ball of $JT^*_{\mathcal{F}_*}$ is the norm closed convex hull of its extreme points. Since the set $\mathcal{F}_*$ is the norming set of the space $JT_{\mathcal{F}_*}$ we have that $B_{JT^*_{\mathcal{F}_*}} = \overline{\mathrm{conv}(\mathcal{F}_*)}^{w^*}$ hence $\mathrm{Ext}(B_{JT^*_{\mathcal{F}_*}}) \subset \overline{\mathcal{F}_*}^{w^*}$. We thus get that $JT^*_{\mathcal{F}_*} = \overline{\mathrm{span}}(\overline{\mathcal{F}_*}^{w^*})$.

We observe that

$$\begin{aligned}\overline{\mathcal{F}_{2,s}}^{w^*} \quad &= \quad F_0 \cup \big\{ \sum_{i=1}^{d} a_i x_i^* : \ \sum_{i=1}^{d} a_i^2 \leq 1, \ (x_i^*)_{i=1}^{d} \text{ are } \sigma_{\mathcal{F}} \text{ special functionals} \\ &\qquad \text{with } (\mathrm{ind}(x_i^*))_{i=1}^{d} \text{ pairwise disjoint and } \min\operatorname{supp} x_i^* \geq d \big\},\end{aligned}$$

$$\begin{aligned}\overline{\mathcal{F}_{2}}^{w^*} \quad &= \quad F_0 \cup \big\{ \sum_{i=1}^{\infty} a_i x_i^* : \ \sum_{i=1}^{\infty} a_i^2 \leq 1, \ (x_i^*)_{i=1}^{\infty} \text{ are } \sigma_{\mathcal{F}} \text{ special functionals} \\ &\qquad \text{with } (\mathrm{ind}(x_i^*))_{i=1}^{\infty} \text{ pairwise disjoint} \big\}.\end{aligned}$$

and

$$\begin{aligned}\overline{\mathcal{F}_{s}}^{w^*} \quad &= \quad F_0 \cup \big\{ \sum_{i=1}^{d} \varepsilon_i x_i^* : \ \varepsilon_i \in \{-1,1\}, \ (x_i^*)_{i=1}^{d} \text{ are } \sigma_{\mathcal{F}} \text{ special functionals} \\ &\qquad \text{with } (\mathrm{ind}(x_i^*))_{i=1}^{d} \text{ pairwise disjoint and } \min\operatorname{supp} x_i^* \geq d \big\},\end{aligned}$$

The first and third equality follow easily. For the second the arguments are similar to Lemma 8.4.5 of [Fa].

The first part of the proposition for the cases $\mathcal{F}_* = \mathcal{F}_{2,s}$ or $\mathcal{F}_* = \mathcal{F}_s$ follows directly while for the case $\mathcal{F}_* = \mathcal{F}_2$ it is enough to observe that $\| \sum_{i=1}^{\infty} a_i x_i^* \|_{JT^*_{\mathcal{F}_2}} \leq \big(\sum_{i=1}^{\infty} a_i^2 \big)^{1/2}$ for every $g = \sum_{i=1}^{\infty} a_i x_i^* \in \mathcal{F}_2$. Therefore

$$JT^*_{\mathcal{F}_*} = \overline{\mathrm{span}}(\{e_n^* : \ n \in \mathbb{N}\} \cup \{b^* : \ b \text{ branch}\}).$$

From the first part of the proposition, to show that the basis $(e_n)_{n\in\mathbb{N}}$ is weakly null, it is enough to show that $b^*(e_n) \overset{n\to\infty}{\longrightarrow} 0$ for every branch b. But if $b = (f_1, f_2, f_3, \ldots)$ is an arbitrary branch then the sequence $k_n = \mathrm{ind}(f_n)$ is strictly increasing and hence, since $\|f_n\|_\infty \leq \tau_{k_n}$, the conclusion follows. □

Remark B.16. Let $(F_j)_{j=0}^{\infty}$ be a JTG family (Definition B.1). If τ is a subfamily of the family of finite $\sigma_{\mathcal{F}}$ special functionals such that $F \subset \tau$ and $Ex^* \in \tau$ for every $x^* \in \tau$ and interval E of $\mathbb{N}$, then, setting

$$\begin{aligned}\mathcal{F}_{\tau,s} \quad &= \quad \{ \sum_{i=1}^{d} \varepsilon_i x_i^* : \ \varepsilon_1, \ldots, \varepsilon_d \in \{-1,1\}, \ x_1^*, \ldots, x_d^* \in \tau \\ &\qquad \text{with } \mathrm{ind}(x_i^*)_{i=1}^{d} \text{ pairwise disjoint and } \min\operatorname{supp} x_i^* \geq d, \ d \in \mathbb{N}\}.\end{aligned}$$

the space $JT_{\mathcal{F}_{\tau,s}}$, which is defined to be the completion of $(c_{00}(\mathbb{N}) \ , \| \ \|_{\mathcal{F}_{\tau,s}})$, is also c_0 saturated.

Theorem B.17. There exists a Banach space X with a weakly null Schauder basis $(e_n)_{n\in\mathbb{N}}$ such that X is ℓ_2 saturated (c_0 saturated) and for every $M \in [\mathbb{N}]$ the space $X_M = \overline{\text{span}}\{e_n : n \in M\}$ has nonseparable dual.

A similar result has been also obtained by E. Odell in [O] using a different approach.

Proof. Let $(F_j)_{j=0}^{\infty}$ be the Maurey-Rosenthal JTG family (Example B.2 (i)). As we have seen the space $X = JT_{\mathcal{F}_2}$ (Definition B.5) has a normalized weakly null Schauder basis $(e_n)_{n\in\mathbb{N}}$ (Proposition B.15) and it is ℓ_2 saturated (Theorem B.14).

Let now $M \in [\mathbb{N}]$. We inductively construct $(x_a, f_a, j_a)_{a\in\mathcal{D}}$, where $\mathcal{D}$ is the dyadic tree and the induction runs on the lexicographical order of $\mathcal{D}$, such that the following conditions are satisfied:

(i) For every $a \in \mathcal{D}$ there exists $F_a \subset M$ with $\#(F_a) = k_{j_a}$ such that $x_a = \frac{1}{\sqrt{k_{j_a}}} \sum_{i\in F_a} e_i$ and $f_a = \frac{1}{\sqrt{k_{j_a}}} \sum_{i\in F_a} e_i^*$.

(ii) $j_\emptyset \in \Xi_1$ with $j_\emptyset \geq 2$ while for $a \in \mathcal{D}$, $a \neq \emptyset$, $j_a = \sigma_{\mathcal{F}}((f_\beta)_{\beta<a})$.

(iii) If $a <_{lex} \beta$ then $F_a < F_\beta$.

Our construction yields that for every branch b of the dyadic tree the sequence $(f_a)_{a\in b}$ is a $\sigma_{\mathcal{F}}$ special sequence. Hence the w^* sum $g_b = \sum_{a\in b} f_a$ is a member of $\overline{\mathcal{S}}^{w^*}$ and thus it belongs to the unit ball of $JT_{\mathcal{F}_2}$. We shall show that $\|g_b|_{X_M} - g_{b'}|_{X_M}\|_{X_M^*} \geq \frac{1}{2}$ for infinite branches $b \neq b'$ of the dyadic tree.

We first observe that for every $a \in \mathcal{D}$ and $f \in F_j$ we have that $|f(x_a)| \leq \min\{\frac{\sqrt{k_j}}{\sqrt{k_{j_a}}}, \frac{\sqrt{k_{j_a}}}{\sqrt{k_j}}\}$. Thus, if $g = \sum_{i=1}^{d} a_i x_i^* \in \mathcal{F}_2$ (Definition B.5), then

$$|g(x_a)| \leq \sum_{i=1}^{d} |x_i^*(x_a)| \leq \sum_{j<j_a} \frac{\sqrt{k_j}}{\sqrt{k_{j_a}}} + 1 + \sum_{j>j_a} \frac{\sqrt{k_{j_a}}}{\sqrt{k_j}} \leq 1 + 1 = 2.$$

We conclude that $x_a \in X_M$ with $\|x_a\|_{\mathcal{F}_2} \leq 2$.

Therefore, if $b \neq b'$ are infinite branches of $\mathcal{D}$ and $a \in b \setminus b'$ then

$$\|g_b|_{X_M} - g_{b'}|_{X_M}\|_{X_M^*} \geq \frac{(g_b - g_{b'})(x_a)}{\|x_a\|_{\mathcal{F}_2}} \geq \frac{f_a(x_a)}{2} = \frac{1}{2}.$$

The c_0 saturated space of the statement is the space $X = JT_{\mathcal{F}_{2,s}}$ and the proof is the same. □

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