A CLASS OF BANACH SPACES WITH FEW NON STRICTLY SINGULAR OPERATORS

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The original motivation for this paper is based on the natural question left open by the Gowers-Maurey solution of the unconditional basic sequence problem for Banach spaces ([12]). Recall that Gowers and Maurey have constructed a Banach space X with a Schauder basis $(e_n)_n$ but with no unconditional basic sequence. Thus, while every infinite dimensional Banach space contains a sequence $(x_n)_n$ which forms a Schauder basis for its closure $Y = \overline{\langle x_n\rangle_n}$, meaning that every vector of Y has a unique representation $\sum_n a_n x_n$, one may not be able to get such $(x_n)_n$ such that the sums $\sum_n a_n x_n$ converge unconditionally whenever they converge. The fundamental role of Schauder basis and the fact that the notion is very much dependent on the order lead to the natural variation of the notion, the definition of transfinite Schauder basis $(x_\alpha)_{\alpha<\gamma}$, where vectors of X have a unique representations as sums $\sum_{\alpha<\gamma} a_\alpha x_\alpha$. In fact, as it will be clear from some results in this paper, considering a transfinite Schauder basis, even if one knows that X has an ordinary Schauder basis, can be an advantage. Thus, the natural question which originated the research of this paper asks whether one can have Banach spaces with long (even of uncountable length) Schauder bases but with no unconditional basic sequence. There is actually a more fundamental reason for asking this question. As noticed originally by W. B. Johnson, the Gowers-Maurey space X is hereditarily indecomposable which in particular yields that the space of operators on X is very small in the sense that every bounded linear operator on X can be written as $\lambda \mathrm{Id}_X + S$, where S is a strictly singular operator. On the other hand, if X has a transfinite Schauder basis $(e_\alpha)_{\alpha<\gamma}$ of length, say, $\gamma = \omega^2$, it could no longer have so small an operator space as projections on infinite intervals $\overline{\langle e_\alpha\rangle_{\alpha\in I}}$ are all (uniformly) bounded. Thus one would like to find out the amount of control on the space of non strictly singular operators that is possible in this case. In fact, our solution of the transfinite variation of the unconditional basic sequence problem has led us to many other new questions of this sort, has forced us to introduce several new methods to this area, and has revealed several new phenomena that could have been perhaps difficult to discover by working only in the context of ordinary Schauder bases.

To see the necessity for a new method we repeat that our first goal here is to construct a Banach space $\mathfrak{X}_{\omega_1}$ with a transfinite Schauder basis $(e_\alpha)_{\alpha<\omega_1}$ with no unconditional basic sequence as well as to understand its separable initial segments $\mathfrak{X}_\gamma = \overline{\langle e_\alpha\rangle_{\alpha<\gamma}}$. The original Gowers-Maurey method for preventing unconditional basic sequences is to force the unconditional constants of initial finite-dimensional subspaces, according to the fixed Schauder basis, grow to infinity. Since initial finite-dimensional subspaces according to our transfinite Schauder

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basis $(e_\alpha)_{\alpha<\omega_1}$ are far from exhausting the whole space their method will not work here. It turns out that in order to impose the conditional structure on our space(s) $\mathfrak{X}_\gamma$ ($\gamma \leq \omega_1$) we needed to import a tool from another area of mathematics, a rather canonical semi-distance function ϱ on the space ω_1 of all countable ordinals ([26]). What ϱ does in our context here is to essentially identify the structure of finite-dimensional subspaces of various $\mathfrak{X}_\gamma$'s which globally are of course very much different, since for example $\mathfrak{X}_\omega$ is hereditarily indecomposable while, say, $\mathfrak{X}_{\omega^2}$ has a rich space of non-strictly singular operators.

After solving this initial problem we went on and tried to show that every bounded linear operator T on a given $\mathfrak{X}_\gamma$ is a sum of a diagonal operator D_T and a strictly singular one. There are natural candidates for D_T which would share the eigenvalues of T and have the property that $T - D_T$ is strictly singular. The problem is to show that D_T is a bounded operator. This forced us to a variation on the notion of ϱ-function by adding to it certain universality property. To see the need for this, suppose we are given a finitely supported vector x such that $\|D_T x\|$ is very large in comparison with $\|Tx\|$. The vector x has a natural decomposition $x = x_1 + \cdots + x_n$ such that $D_T x = \lambda_1 x_1 + \cdots + \lambda_n x_n$ where λ_i's are eigenvalues of T. The universality of ϱ guarantees that x_i's can all be simultaneously moved (keeping the discrepancy between $\|D_T x\|$ and $\|Tx\|$) to be almost equal to eigenvectors with eigenvalues λ_i's giving us an impossibility. This also gives us a new phenomenon, unprecedented in this area, that every finite dimensional subspace of some $\mathfrak{X}_\gamma$ can be moved by an $(4+\varepsilon)$-isomorphism to essentially any region of any other $\mathfrak{X}_\delta$.

Our attempt to extend the control of operators to arbitrary subspaces of $\mathfrak{X}_{\omega_1}$ has led us to a new phenomenon which a priori could have been discovered before since it already has a solid basis in an old paper of Maurey-Rosenthal ([20]). What we discovered is that each $\mathfrak{X}_\gamma$ has an associated James-like space J_{T_0} which is minimally and canonically finitely block represented in $\mathfrak{X}_\gamma$ and which is responsible for essentially all of its conditional and unconditional geometry, including the complete structure of the corresponding space of bounded non-strictly singular operators. In retrospect, what Maurey-Rosenthal [20] have done in their attempt to solve the unconditional basic sequence problem is to produce a space X with a Schauder basis $(e_n)_n$ such that every subsequence $(e_{n_k})_k$ finitely block represents J_{c_0}, a fact which then they used to show that no subsequence of $(e_n)_n$ is unconditional. The finite representability of J_{T_0} and the global control of block sequences provided by ϱ gives us a complete picture of the space of bounded non-strictly singular operators defined not only on $\mathfrak{X}_\gamma$ ($\gamma \leq \omega_1$) but also on their arbitrary subspaces. For example, we show that the space of all bounded non-strictly singular operators on a given $\mathfrak{X}_\gamma$ is naturally isomorphic to the dual of the corresponding James-like space J_{T_0}. We also discover subspaces X of $\mathfrak{X}_\gamma$ such that the non-strictly singular part of the operator space $\mathcal{L}(X, \mathfrak{X}_\gamma)$ is quite rich but on the other hand every bounded operator $T : X \to X$ is a strictly singular perturbation of a scalar multiple of the identity. Another new phenomenon we found are hereditarily indecomposable subspaces of $\mathfrak{X}_{\omega_1}$ that are asymptotic versions of themselves.

We now pass to a more detailed presentation of the specific results of this paper. The first section concerns extensions of some standard facts about Schauder basic sequences to the transfinite case. For example we show that every subspace Y of a space X with a transfinite basis contains a further subspace Z isomorphic to a block subspace of X. We should point out that this result is weaker than the corresponding results for Schauder bases. This causes some problems when one tries to extend standard constructions into the transfinite case. For example, if one considers the transfinite version of the Schlumprecht space, or more generally spaces of the form $T_\gamma[(1/m_j, n_j)_j]$, one runs into difficulties when trying to prove arbitrarily distortion. We overcome this by adding a property to the basis $(x_\alpha)_{\alpha<\gamma}$ which ensures that the block sequences approximate in a strong sense the subspaces of the space $\mathfrak{X}_{\omega_1}$. This condition permits us to show that the spaces $\mathfrak{X}_{\omega_1}$ and $\mathfrak{X}^u_{\omega_1}$ are arbitrarily distortable. We also give a characterization of reflexivity analogous to the classical one due to James [15].

The second section is mainly devoted to the definition of the norming set K_{ω_1} of the maximal space $\mathfrak{X}_{\omega_1}$. This set is a subset of the norming set of the transfinite mixed Tsirelson space $T_{\omega_1}[(1/m_j, n_j)]$. The norm can also be described by the following implicit formula, for $x \in c_{00}(\omega_1)$,

$$\|x\|_* = \max\{\|x\|_\infty, \sup_j\{\sup \frac{1}{m_{2j}} \sum_{i=1}^{n_{2j}} \|E_i x\|_*,\ E_1 < \cdots < E_{n_{2j}}\}\vee$$
$$\vee \sup\{\frac{1}{m_{2j+1}} \sum_{i=1}^{n_{2j+1}} \phi_i(x)\ :\ \{\phi_i\}_{i=1}^{n_{2j+1}} \text{ is a } n_{2j+1}\text{-special sequence}\}\}.$$

This definition shares the same components with the corresponding definition of the separable hereditarily indecomposable spaces. The crucial difference concerns the definition of n_{2j+1}-*special sequences.* For this we introduce a new coding σ_ϱ based on a ϱ function which while it cannot be one-to-one anymore it does provide a tree-like interference between pairs of special sequences sufficient to impose a strong conditional structure on $\mathfrak{X}_{\omega_1}$.

The aim of the third section is to explain how the new $\varrho-$coding is used in proving some of the basic properties of the space $\mathfrak{X}_{\omega_1}$. Thus, postponing the proofs of some estimations for the next section, we show that block subsequences of $(e_\alpha)_{\alpha<\omega_1}$ generate hereditarily indecomposable subspaces. Section four contains the basic estimations which are analogous to the $\omega-$case. We also show that $\mathfrak{X}_{\omega_1}$ is reflexive. The fifth section contains the study of the bounded linear operators. As we have mentioned above many of the results are based on the finite representability of the James-like space J_{T_0} in the transfinite block subsequences of $\mathfrak{X}_{\omega_1}$. There are two ways to define J_{T_0}. The first is the Bellenot-Haydon-Odell definition ([8]) of the Jamesfication of the mixed Tsirelson space $T_0 = T[(1/m_{2j}, n_{2j})_j]$ and the second is the Tsirelson-like space $T[G, (1/m_{2j}, n_{2j})_j]$ with $G = \{\chi_I :\ I \text{ interval of } \mathbb{N}\}$. The space J_{T_0} is quasi reflexive and for every set of ordinals A the space $J_{T_0}(A)$ is defined similarly to [9]. The study of $J_{T_0}(A)$ and the finite representability of J_{T_0} in $\mathfrak{X}_{\omega_1}$ are contained in the first two subsections of section five. The remaining subsections are devoted to the study of the spaces of operators. The central notion of *step diagonal operator* is defined as follows. Let X be a subspace of $\mathfrak{X}_{\omega_1}$ generated by a transfinite block sequence $(x_\alpha)_{\alpha<\gamma}$. A bounded linear diagonal operator $D : X \to X$ is a step diagonal operator if $\lambda_\alpha = \lambda_\beta$ for all α, β with $\alpha \le \beta < \alpha + \omega \le \gamma$. For example, if $\gamma = \omega$ then $D = \lambda \mathrm{Id}_X$ and if $\gamma = \omega^2$ then $D = \sum_n \lambda_n \mathrm{Id}_n$ where Id_n is the identity of $X_n = \overline{\langle (x_\alpha)_{\alpha\in[\omega(n-1),\omega n)} \rangle}$. We prove the following result.

Theorem. *There exists a universal constant $C > 0$ such that for every countable limit ordinal γ there is a set of ordinals A_γ such that: For every transfinite block sequence $(x_\alpha)_{\alpha<\gamma}$ in $\mathfrak{X}_{\omega_1}$, the algebra $\mathcal{D}(\overline{\langle (x_\alpha)_{\alpha<\gamma}\rangle})$ of the step diagonal operators is C-isomorphic to $J^*_{T_0}(A_\kappa)$.*

There are several consequences of this theorem: It follows readily that the structure of $\mathcal{D}(X)$ for X generated by a transfinite block sequence $(x_\alpha)_{\alpha<\gamma}$ depends only on the ordinal γ. The dimension of $\mathcal{D}(X)$ is equal to the cardinality of the set A_γ, and for every $D \in \mathcal{D}(X)$ and $\varepsilon > 0$ there is an operator of the form $\sum_{i=1}^n \lambda_i P_{I_i}$, with $\{I_i\}_{i=1}^n$ intervals of γ, which ε-approaches D. Furthermore the following holds.

Theorem. *There exists a universal constant $C > 0$ such that for every subspace X of $\mathfrak{X}_{\omega_1}$ generated by transfinite block sequence $(x_\alpha)_{\alpha<\gamma}$ of $\mathfrak{X}_{\omega_1}$ and every $T \in \mathcal{L}(X, \mathfrak{X}_{\omega_1})$ we have:*

(i) *$T = D_T + S_T$ where $D_T \in \mathcal{D}(X)$, $\|D_T\| \le C\|T\|$ and $S : X \to \mathfrak{X}_{\omega_1}$ is strictly singular.*
(ii) *Every $D \in \mathcal{D}(X)$ is extendable to a $\tilde{D} \in \mathcal{D}(\mathfrak{X}_{\omega_1})$ with $\|\tilde{D}\| \le C\|D\|$.*
(iii) *$\mathcal{L}(X, \mathfrak{X}_{\omega_1}) \cong J^*_{T_0}(A_\kappa) \oplus \mathcal{S}(X, \mathfrak{X}_{\omega_1})$.*

We also introduce the notion of asymptotically equivalent subspaces of $\mathfrak{X}_{\omega_1}$ which permits us to extend part (iii) of the above theorem to arbitrary subspaces of $\mathfrak{X}_{\omega_1}$. Namely for every subspace X of $\mathfrak{X}_{\omega_1}$ there exists a set of ordinals A_X such that $\mathcal{L}(X, \mathfrak{X}_{\omega_1}) \cong J^*_{T_0}(A_X) \oplus \mathcal{S}(X, \mathfrak{X}_{\omega_1})$. We are not able however to provide a sufficient description of $\mathcal{L}(X)$ for an arbitrary subspace X of $\mathfrak{X}_{\omega_1}$. What we have noticed is that in general $\mathcal{L}(X, \mathfrak{X}_{\omega_1})/\mathcal{S}(X, \mathfrak{X}_{\omega_1}) \ncong \mathcal{L}(X)/\mathcal{S}(X)$. For strictly singular operators on $\mathfrak{X}_{\omega_1}$ we give the following characterization.

Theorem. *An operator $S : \mathfrak{X}_{\omega_1} \to \mathfrak{X}_{\omega_1}$ is strictly singular iff the sequence $(\|S(e_\alpha)\|)_{\alpha<\omega_1} \in c_0(\omega_1)$.*

Corollary. *Every $T \in \mathcal{L}(\mathfrak{X}_{\omega_1})$ has the form $T = \lambda \mathrm{Id}_{\mathfrak{X}_{\omega_1}} + D + S$ where $D \in \mathcal{D}(X_\gamma)$ for some $\gamma < \omega_1$, and S is strictly singular. In particular $T = \lambda \mathrm{Id}_{\mathfrak{X}_{\omega_1}} + Q$ where Q has separable range.*

We mention that nonseparable spaces X such that all $T \in \mathcal{L}(X)$ are of the form $\lambda \mathrm{Id}_X + Q$ with the range of Q separable have been constructed before in [23], [24] and [28]. However, those constructions are quite different from ours as they, in particular, offer no information about operators on separable subspaces of the resulting space X.

Furthermore, we show that for I, J disjoint intervals of ω_1 the spaces $\mathfrak{X}_I$ and $\mathfrak{X}_J$ are totally incomparable and the space $\mathfrak{X}_{\omega_1}$ is arbitrarily distortable. Moreover, modulo strictly singular perturbations, the space $\mathfrak{X}_{\omega_1}$ admits a unique resolution of the identity. Out of the rich sources of examples of subspaces of $\mathfrak{X}_{\omega_1}$ with interesting spaces of operators we mention the following

Theorem. *There exists a separable reflexive Banach space $\mathfrak{X}$ admitting an infinite dimensional Schauder decomposition $\mathfrak{X} = \bigoplus_n \mathfrak{X}_n$ such that, denoting by $\mathcal{D}(\mathfrak{X})$ the class of bounded operators $D : \mathfrak{X} \to \mathfrak{X}$ with the property $D|_{\mathfrak{X}_n} = \lambda_n \mathrm{Id}_{\mathfrak{X}_n}$ for all n, the following hold:*

(i) *$\mathcal{L}(\mathfrak{X}) \cong \mathcal{D}(\mathfrak{X}) \oplus \mathcal{S}(\mathfrak{X}) \cong J^*_{T_0} \oplus \mathcal{S}(\mathfrak{X})$.*
(ii) *For every subspace X of $\mathfrak{X}$ there exists $A \subseteq \mathbb{N}$ which is either an initial finite interval or is equal to $\mathbb{N}$ such that $\mathcal{L}(X, \mathfrak{X}) \cong J^*_{T_0}(A) \oplus \mathcal{S}(X, \mathfrak{X})$.*
(iii) *There is a subspace X of $\mathfrak{X}$ such that $\mathcal{L}(X) \cong \langle I_X \rangle \oplus \mathcal{S}(X)$ while $\mathcal{L}(X, \mathfrak{X}) \cong J^*_{T_0} \oplus \mathcal{S}(X, \mathfrak{X})$.*

For example, the space $\mathfrak{X} = \mathfrak{X}_{\omega^2}$ has all these properties. It is worth pointing out that $D(\mathfrak{X})$ is a natural class of operators which behaves similarly to the class of operators of the form $\lambda I + K$ with K compact diagonal. For example if $x_n \in \mathfrak{X}_n$ with $\|x_n\| = 1$ and $X = \overline{\langle (x_n)_n \rangle}$ then for every $D \in \mathcal{D}(\mathfrak{X})$ we have that $D|_X = \lambda I + K$. The isomorphism between $\mathcal{D}(\mathfrak{X})$ and $J^*_{T_0}$ endows $J^*_{T_0}$ with an equivalent norm under which $J^*_{T_0}$ with the pointwise multiplication becomes a commutative Banach algebra. This should be compared with results from [1].

Sections six and seven concern two new properties that can be simultaneously imposed on a ϱ-function and the resulting properties of $\mathfrak{X}_{\omega_1}$. First, we present a construction of a universal ϱ-function where universality roughly speaking means that for every infinite interval I of ω_1 the finite $\varrho-$closed subsets of I realize all isomorphism types of finite submodels of all possible ϱ-functions. As we have mentioned before, our initial motivation for introducing the universal ϱ-function was to understand the structure of $\mathcal{L}(\mathfrak{X}_{\omega_1})$. However it turns out that using the $\varrho-$coding with a universal ϱ we obtain some new properties on $\mathfrak{X}_{\omega_1}$ which have their own interest, even for the space $\mathfrak{X}_\omega = \overline{\langle e_n : n < \omega \rangle}$. Indeed $\mathfrak{X}_{\omega_1}$ admits a nearly subsymmetric transfinite basis and moreover $\mathfrak{X}_\omega$, which is an hereditarily indecomposable space, is an asymptotic version of itself [19]. The results concerning subsymmetric transfinite sequences and asymptotic versions are presented in section seven. Section six also contains the construction of smooth ϱ-functions and the following result. If the coding σ_ϱ is based on a smooth ϱ-function then every countable ordinal $\gamma < \omega_1$ can be re-ordered as $(\alpha_n)_{n<\omega}$ such that $(e_{\alpha_n})_{n<\omega}$ defines a Schauder basis of the space X_γ. Section eight contains a unified approach of the proof of two important results, the basic inequality and the nontrivial direction of the finite representability of J_{T_0}. Their proofs share some common features, and so we attempt to develop a general theory that includes both results and that could be useful elsewhere. The last section is devoted to the unconditional counterpart of the space $\mathfrak{X}_{\omega_1}$ denoted by $\mathfrak{X}^u_{\omega_1}$. The relation of $\mathfrak{X}^u_{\omega_1}$, which is a space with an unconditional basis $(e_\alpha)_{\alpha<\omega_1}$, with the space $\mathfrak{X}_{\omega_1}$ is same as that of Gowers-Maurey space with Gowers unconditional space [11]. We study the structure of $\mathcal{L}(\mathfrak{X}^u_{\omega_1})$ and the structure the subspaces of $\mathfrak{X}^u_{\omega_1}$.

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1. Transfinite basic sequences

The first section concerns the presentation of some preliminary results related to transfinite (Schauder) bases. We recall one of the equivalent formulations of their definition. For a detailed presentation we refer the reader to [25].

Definition 1.1. Let X be a Banach space, and γ be an ordinal number.
1. A total family $(x_\alpha)_{\alpha<\gamma}$ of elements of X (i.e., a family such that $X = \overline{\langle x_\alpha \rangle_{\alpha<\gamma}}$) is said to be a *transfinite basis* if there exists a constant $C \geq 1$ such that for every interval I of γ the naturally defined map on the linear span of $(x_\alpha)_{\alpha<\gamma}$

$$\sum_{\alpha<\gamma} \lambda_\alpha x_\alpha \mapsto \sum_{\alpha\in I} \lambda_\alpha x_\alpha$$

extends to a bounded projection $P_I : X \to X_I = \overline{\langle x_\alpha \rangle_{\alpha\in I}}$ of norm at most C.

2. A transfinite basis $(x_\alpha)_{\alpha<\gamma}$ of X is said to be *bimonotone* if for each interval I of γ, the corresponding projection P_I has norm 1.
3. A transfinite basis $(x_\alpha)_{\alpha<\gamma}$ of X is said to be *unconditional* if there exists a constant $C \geq 1$ such that for all subsets A of γ, the corresponding P_A has norm at most C.
4. A transfinite basis $(x_\alpha)_{\alpha<\gamma}$ of X is said to be *1-subsymmetric* if for every $n \in \mathbb{N}$, every $\alpha_1 < \alpha_2 < \cdots < \alpha_n < \gamma$ and every $(\lambda_i)_{i=1}^n \in \mathbb{R}^n$, $\|\sum_{i=1}^n \lambda_i x_i\| = \|\sum_{i=1}^n \lambda_i x_{\alpha_i}\|$.

Remark 1.2. 1. As in the case of the usual Schauder basis (i.e., $\gamma = \omega$) the above definition is equivalent to the fact that each $x \in X$ admits a unique representation as $\sum_{\alpha<\gamma} \lambda_\alpha x_\alpha$, where the convergence of these series is recursively defined.
2. The definition of $\sum_{\alpha<\gamma} \lambda_\alpha x_\alpha$ easily yields that for each convergent series $\sum_{\alpha<\gamma} \lambda_\alpha x_\alpha$ with $(x_\alpha)_{\alpha<\gamma}$ a bounded family, the sequence of coefficients $(\lambda_\alpha)_{\alpha<\gamma}$ belongs to $c_0(\gamma)$. Furthermore, for every $\varepsilon > 0$ there exists a finite subset F of γ such that $\|\sum_{\alpha\notin F} \lambda_\alpha x_\alpha\| < \varepsilon$.
3. For every transfinite basis $(x_\alpha)_{\alpha<\gamma}$ the dual basis $(x^*_\alpha)_{\alpha<\gamma}$ is also well defined. Just like the usual Schauder bases, $(x^*_\alpha)_{\alpha<\gamma}$ is a w^*-total subset of X^* and each x^* in X^* has a unique representation of the form $\sum_{\alpha<\gamma} x^*(x_\alpha) x^*_\alpha$ where the series is w^*-convergent.
4. If $(x_\alpha)_{\alpha<\gamma}$ is a transfinite basis for the space $(X, \|\cdot\|)$, then there exists an equivalent norm $|||\cdot|||$ on X such that $(x_\alpha)_{\alpha<\gamma}$ is a bimonotone basis for the space $(X, |||\cdot|||)$. This norm is defined by $|||x||| = \sup\{\|P_I(x)\| \,:\, I \text{ interval of } \gamma\}$.

In the sequel, for every ordinal γ we shall denote by $c_{00}(\gamma)$ the vector space of all sequences $(\lambda_\alpha)_{\alpha\in\gamma}$ of real numbers such that the set $\{\alpha < \gamma \,:\, \lambda_\alpha \neq 0\}$ is finite. We also denote by $(e_\alpha)_{\alpha<\gamma}$ the natural Hamel basis of $c_{00}(\gamma)$. It is an easy observation that every space X with a transfinite basis $(x_\alpha)_{\alpha<\gamma}$ is isometric to the completion of $c_{00}(\gamma)$ endowed with an appropriate norm. Moreover if K is a subset of $c_{00}(\gamma)$ with the properties (a) $\{e^*_\alpha\}_{\alpha<\gamma} \subseteq K$ and (b) for every $\phi \in K$, $\|\phi\|_\infty \leq 1$ and for every interval I of γ, the restriction $\phi_I = \phi \cdot \chi_I$ of ϕ to I is also a member of K, then the norm defined on $c_{00}(\gamma)$ by

$$\|x\|_K = \sup\{|\phi(x)| = \langle \phi, x\rangle \,:\, \phi \in K\}$$

has $(e_\alpha)_{\alpha<\gamma}$ as a transfinite bimonotone basis for the completion of $(c_{00}(\gamma), \|\cdot\|_K)$.

Fix X with a transfinite basis $(x_\alpha)_{\alpha<\gamma}$. The support $\operatorname{supp} x$ of $x \in X$ is the set $\{\alpha < \gamma \,:\, x^*_\alpha(x) \neq 0\}$. For a given interval $I \subseteq \gamma$, let $X_I = P_I X$, and for $\alpha < \gamma$, let $X_\alpha = X_{[0,\alpha)}$. For $x, y \in X$ finitely supported, we write $x < y$ to denote that $\max \operatorname{supp} x < \min \operatorname{supp} y$.

A sequence $(y_\alpha)_{\alpha<\xi}$ is called a transfinite block subsequence of $(x_\alpha)_{\alpha<\gamma}$ if and only if for all $\alpha < \xi$, y_α is finitely supported and for all $\alpha < \beta < \xi$, $y_\alpha < y_\beta$. Notice that a transfinite block subsequence of a transfinite basis is always a transfinite basis of its closed linear span.

Fix two Banach spaces X and Y. A bounded operator $T : X \to Y$ is an *isomorphism* iff TX is closed and T is 1-1. T is called *strictly singular* if it is not an isomorphism when restricted to any infinite dimensional closed subspace of X (i.e., for all infinite dimensional closed subspace X' of X, either TX' is not closed or $T|X'$ is not $1-1$). This is equivalent to say that for all infinite dimensional closed subspace Y of X and $\varepsilon > 0$, there is an infinite dimensional closed subspace Y' of Y such that $\|T|Y'\| \leq \varepsilon$.

It is well known that most of the structure of the infinite dimensional closed subspaces of a separable Banach space X with a basis $(x_n)_n$ is described by its block sequences. Namely that for every infinite dimensional closed subspace Y of X and every $\varepsilon > 0$ there exists a normalized sequence in Y and a block sequence $(w_n)_n$ of $(x_n)_n$ which are $1+\varepsilon$-equivalent. The method used for the proof of this result is called the *gliding hump* argument ([17]). This result is not extendable in the case of the transfinite block sequences. For example, consider a biorthogonal basis $(x_\alpha)_{\alpha<\omega\cdot 2}$ of a Hilbert space and let Y be the subspace generated by the sequence $(x_n + x_{\omega+n})_n$.

We now describe how block sequences are connected to subspaces in the transfinite case.

Proposition 1.3. *Let $(x_\alpha)_{\alpha<\gamma}$ be a transfinite basis of X and Y an infinite dimensional closed subspace X. Then there exists a $\lambda \leq \gamma$ and a closed subspace Z of Y such that*
1. $P_\lambda : Z \to X_\lambda$ is an isomorphism.
2. For every $\varepsilon > 0$ there exists a semi-normalized block sequence $(w_n)_n$ in X_λ and a normalized sequence $(z_n)_n$ in Z such that $\sum_n \|P_\lambda z_n - w_n\| < \varepsilon$.
3. There exists a subspace Z' of Z isomorphic to a block subspace of X.
4. If we additionally assume that Y has a Schauder basis $(y_n)_n$, then the sequence $(z_n)_n$ in 2. can be chosen to be a block sequence of $(y_n)_n$.

Proof. We assume that $(x_\alpha)_{\alpha<\gamma}$ is a bimonotone basis. Let

$$\beta_0 = \min\{\beta : P_\beta : Y \to X_\beta \text{ is not strictly singular}\}. \tag{1}$$

Let us show that $\lambda = \beta_0$ is the required ordinal. Notice that β_0 has to be necessarily a limit ordinal. Since P_{β_0} is not strictly singular on Y, there exists a subspace Z of Y such that $P_{\beta_0} : Z \to X_{\beta_0}$ is an isomorphism. On the other hand for every $\gamma < \beta_0$, $P_\gamma : Y \to X_\gamma$ is strictly singular hence for every $\varepsilon > 0$ and every subspace Z' of Z there exists $W \hookrightarrow Z'$[1] such that $\|P_\gamma|W\| < \varepsilon$. Now we are ready to apply a modified gliding hump argument to obtain $(z_n)_n$, $(w_n)_n$ as they are required in 2. Indeed for a given ε we choose $(\varepsilon_n)_n$ such that $\varepsilon_n > 0$, $\sum \varepsilon_n < \varepsilon/4$. We choose a normalized $z_1 \in Z$. Since β_0 is a limit ordinal, there must exist $\gamma_1 < \beta_0$ such that $\|P_{[\gamma_1,\beta_0)} z_1\| < \varepsilon_1$. Hence setting $w_1 = P_{\gamma_1} z_1$ we have that $\|w_1 - P_{\beta_0} z_1\| < \varepsilon_1$. Since $P_{\gamma_1} : Z \to X_{\gamma_1}$ is strictly singular there exists a normalized $z_2 \in Z$ with $\|P_{\gamma_1} z_2\| < \varepsilon_2$. Choose $\gamma_2 > \gamma_1$ such that $\|P_{[\gamma_2,\beta_0)} z_2\| < \varepsilon_2$ and set $w_2 = P_{[\gamma_1,\gamma_2)} z_2$. Observe that $\|P_{\beta_0} z_2 - w_2\| < 2\varepsilon_2$ and $w_1 < w_2$. Continuing in this manner we obtain $(z_n)_n$ and $(w_n)_n$ such that for all n, $\|P_{\beta_0} z_n - w_n\| \leq 2\varepsilon_n$, hence

$$\sum_n \|P_{\beta_0} z_n - w_n\| \leq \varepsilon/2. \tag{2}$$

Since we assume that the transfinite basis $(x_\alpha)_{\alpha<\gamma}$ is bimonotone, (2) implies that $(P_{\beta_0} z_n)_n$ and $(w_n)_n$ are equivalent. Property 3. follows from 2., while 4. results from a careful choice of $(z_n)_n$ in 2. $\square$

As we have mentioned in the introduction the manner that block subspaces saturate the subspaces of X is weaker than the corresponding result for spaces X with a basis $(x_n)_n$. In the

[1] We will write $X \hookrightarrow Y$ to denote that X is an infinite dimensional closed subspace of the Banach space Y.

next proposition we provide a sufficient condition which ensures the complete extension of the result from Schauder bases to transfinite Schauder bases fulfilling the additional condition.

Proposition 1.4. *Let $(x_\alpha)_{\alpha<\gamma}$ be a transfinite basis of X. Assume that for all disjoint intervals I, J of γ the spaces X_I and X_J are totally incomparable. Then for every closed infinite dimensional subspace Y of X and every $\varepsilon > 0$ there exist normalized sequences $(y_n)_n$, $(z_n)_n$ such that $(y_n)_n \subseteq Y$, $(z_n)_n$ is a block sequence of $(z_\alpha)_{\alpha<\gamma}$ and $\sum_n \|y_n - z_n\| < \varepsilon$.*

Proof. From Proposition 1.3 there exists a subspace Z of Y and $\lambda \leq \gamma$ such that $P_\lambda : Z \to X_\lambda$ is an isomorphism. Assume that $\lambda < \gamma$ and set $I = [1, \lambda)$ and $J = [\lambda, \gamma)$. Then $P_J : Z \to X_J$ is a strictly singular operator. Hence we may find (w_n), (z_n) as in Proposition 1.3 (2) such that $\sum_n \|P_J(z_n)\| < \varepsilon$ which yields that $\sum_n \|z_n - w_n\| < 2\varepsilon$. □

Definition 1.5. A transfinite basis $(x_\alpha)_{\alpha<\gamma}$ is called *shrinking* iff for all $(\alpha_n)_n \uparrow$, $(x_{\alpha_n})_n$ is shrinking in the usual sense (i.e., $(x^*_{\alpha_n})$ generates in norm the dual of the closed span of $(x_{\alpha_n})_n$).

It is called *boundedly complete* iff for all $(\alpha_n)_n \uparrow$, $(x_{\alpha_n})_n$ is boundedly complete in the usual sense (i.e., for all sequence of scalars $(\lambda_n)_n$, if there is some $C > 0$ such that for all n, $\|\sum_{i=1}^n \lambda_i x_{\alpha_i}\| \leq C$, then $\sum_i \lambda_i x_{\alpha_i}$ converges in norm).

The above definitions are simpler and easier checked than the corresponding ones cited in [25]. The following result is the extension of the well-known James' characterization ([17]) of reflexivity in the general setting of a Banach space with a transfinite basis.

Proposition 1.6. *Let $(x_\alpha)_{\alpha<\gamma}$ be a transfinite basis of X. Then X is reflexive iff $(x_\alpha)_{\alpha<\gamma}$ is shrinking and boundedly complete.*

Proof. The direct implication is consequence of the James' characterization ([15]). The opposite requires the following two Claims:

Claim. *If $(x_\alpha)_{\alpha<\gamma}$ is shrinking, then the biorthogonal basis $(x^*_\alpha)_{\alpha<\gamma}$ generates in the norm topology the dual space X^*.*

Proof of Claim: Assume the contrary. Then there exists $x^* \in X^*$ not in the closed linear span Y of $(x^*_\alpha)_{\alpha<\gamma}$. Set $\beta_0 = \min\{\beta \leq \gamma : P^*_\beta x^* \notin Y\}$. Then $P^*_{\beta_0} x^* \notin Y$ but for all $\gamma < \beta_0$, $P^*_\gamma x^* \in Y$. Therefore there exists an increasing sequence of successive disjoint intervals $I_1 < I_2 < \cdots < I_n < \cdots < \beta_0$ and $\varepsilon > 0$ such that for each $n \in \mathbb{N}$, $P^*_{I_n} x^* \in Y$ and $\|P^*_{I_n} x^*\| \geq \varepsilon$. Observe that if $x^* \in X^*$, $x^* = w^* - \sum_{\alpha<\gamma} \mu_\alpha x^*_\alpha$, where for each $\alpha < \gamma$, $\mu_\alpha = x^*(x_\alpha)$. Moreover if I is an interval of γ such that $P^*_I x^* \in Y$ and $\varepsilon' > 0$, then there is a finite subset $F_{\varepsilon'}$ of I such that $\|y^*_{\varepsilon'} - x^*\| < \varepsilon$ where

$$y^*_{\varepsilon'} = w^* - \sum_{\alpha \in \gamma \setminus I} \mu_\alpha x^*_\alpha + \sum_{\alpha \in F_{\varepsilon'}} \mu_\alpha x^*_\alpha.$$

Using this observation we inductively select finite sets $F_1 \subseteq I_1, \ldots, F_n \subseteq I_n$ such that setting

$$y^*_n = \sum_{i=1}^n \sum_{\alpha \in F_i} \mu_\alpha x^*_\alpha + P_{\beta_0 \setminus \bigcup_{i=1}^n I_n} x^*, \tag{3}$$

we have that

$$\|P^*_{\beta_0}x^* - y^*_n\| < \varepsilon_n < \frac{\varepsilon}{4}. \tag{4}$$

Set $y^* = w^* - \lim_n y^*_n$ and (3) and (4) yield that $\operatorname{supp} y^*_n \subseteq \bigcup_n F_n$ and also $\|P^*_{F_n} y^*\| > \varepsilon/2$. Since each F_n is a finite set we can enumerate $\bigcup_n F_n$ as $(\alpha_n)_n \uparrow$ and clearly y^* yields that the sequence $(x_{\alpha_n})_n$ is not a shrinking Schauder basis, yielding a contradiction. □

Claim. *If $(x_\alpha)_{\alpha<\gamma}$ is boundedly complete, then for every $x^{**} \in X^{**}$, the series $\sum_{\alpha<\gamma} x^{**}(x^*_\alpha)x_\alpha$ converges in norm.*

Proof of Claim: Suppose the contrary and fix $x^{**} \in X^{**}$ but not in X. The proof is similar to the previous one. For each $\alpha < \gamma$, let $\lambda_\alpha = x^{**}x^*_\alpha$ and let

$$\beta_0 = \min\{\beta < \gamma \,:\, P^{**}_\beta x^{**} \notin X\}.$$

Using a similar argument we can choose an increasing sequence $(F_n)_n$ of finite subsets of γ such that $w^* - \sum_{\alpha\in\bigcup_n F_n} \lambda_\alpha x^*_\alpha$ exists and for every n, $\|\sum_{\alpha\in F_n} \lambda_\alpha x_\alpha\| > \varepsilon > 0$. This yields that the sequence $(x_\alpha)_{\alpha\in\bigcup_n F_n}$ is not boundedly complete, a contradiction. □

□

2. Definition of the space $\mathfrak{X}_{\omega_1}$

This section is devoted to the definition of the norm of the space $\mathfrak{X}_{\omega_1}$. This norm will be induced by a set of functionals, denoted by K_{ω_1}, on the space $c_{00}(\omega_1)$. Then $\mathfrak{X}_{\omega_1}$ will be the completion of it. We start with a short presentation of the unconditional frame, which is a mixed Tsirelson space with 1-subsymmetric transfinite basis of a given length γ. The aforementioned set K_{ω_1} will be selected as a subset of B_{Y^*} where Y is the corresponding mixed Tsirelson space.

2.1. **The space $T_\gamma[(1/m_j, n_j)_j]$.** Throughout the paper we fix two infinite sequences $(m_j)_j$, $(n_j)_j$ defined recursively as follows:

1. $m_1 = 2$, and $m_{j+1} = m_j^4$
2. $n_1 = 4$, and $n_{j+1} = (4n_j)^{s_j}$ where $s_j = \log_2 m^3_{j+1}$.

Let γ be an infinite ordinal. Consider the norm $\|\cdot\|_*$ on $c_{00}(\gamma)$ described by the implicit formula

$$\|x\|_* = \max\{\|x\|_\infty, \sup_j \sup \frac{1}{m_j}\sum_{i=1}^{n_j} \|E_i\|_*\},$$

where for $E \subseteq \gamma$, $x \in c_{00}(\gamma)$ Ex denotes the restriction of x to the set E (i.e., $Ex = P_E x = \langle \chi_E, x\rangle$) and the inside supremum is taken over all sequences $E_1 < \cdots < E_{n_j}$ of subsets of γ.

The existence of a norm satisfying the above formula is provided, as the case of Tsirelson space, by an inductive argument (e.g. [17]). It is also easy to see that the usual basis $(e_\alpha)_{\alpha<\gamma}$ of $c_{00}(\gamma)$ defines a 1-subsymmetric and 1-unconditional basis for the space

$$T_\gamma[(m_j^{-1}, n_j)_j] = \overline{(c_{00}(\gamma), \|\cdot\|_*)}.$$

The first variation of the original Tsirelson construction is due to Th. Schlumprecht [22] who introduced the space $S = T_\omega[(1/\log_2(j+1), j)]$ providing the first known example of an arbitrarily distortable Banach space. The space S is one of the key ingredients in the Gowers-Maurey construction [12] of a Banach space with no unconditional basic sequence. The general definition of a mixed Tsirelson space $T_\gamma[(m_j^{-1}, n_j)_j]$ for $\gamma = \omega$ was introduced in [2] using the slightly different notation $T[(\mathcal{A}_{n_j}, 1/m_j)_j]$ which stresses the use of the family $\mathcal{A}_{n_j}$ of all subsets of the index-set (in their case ω) of cardinality at most n_j and indicates the possibility to use some other compact family instead of $\mathcal{A}_{n_j}$ (see e.g. [4] and [7]). Since in this paper we are not going to vary the definition in this direction we suppress the $\mathcal{A}$ as this will give us some notational advantages at some latter points of the paper.

REMARK 2.1. 1. It follows readily from the definition of the norm that for $A \subseteq \gamma$ with order type of A equal to the ordinal λ the space $X_A = \overline{\langle e_\alpha \rangle_{\alpha \in A}}$ is isometric to $T_\lambda[(m_j^{-1}, n_j)_j]$. Therefore granting that $T_\omega[(m_j^{-1}, n_j)_j]$ is reflexive (e.g. [22], [6]) Proposition 1.6 yields that for each γ the space $T_\gamma[(m_j^{-1}, n_j)_j]$ is also reflexive.
2. A possible variation of the norm of $T_\gamma[(m_j^{-1}, n_j)_j]$ is to allow sequences $(E_1, \dots, E_{n_j})$ consisting of disjoint sets (i.e., not necessarily successive). Such spaces are called modified mixed Tsirelson spaces and they are denoted by $T^{\mathcal{M}}[(m_j^{-1}, n_j)_j]$. Schlumprecht has shown that $S^{\mathcal{M}}$ contains ℓ^1 while such spaces have been studied in [3], [18], [4]. The situation for the spaces $T_\omega^{\mathcal{M}}[(m_j^{-1}, n_j)_j]$ remains unclear. Namely, we do not know if there exists a sequence $(q_j^{-1}, l_j)_j$ such that the space $T_\omega^{\mathcal{M}}[(q_j^{-1}, l_j)_j]$ is reflexive and not containing any ℓ_p, $1 < p < \infty$.

There exists an alternative definition of the norm of $T_\gamma[(m_j^{-1}, n_j)_j]$ which is close to the definition of the norm of $\mathfrak{X}_{\omega_1}$. This goes as follows.

Let $L_\gamma \subseteq c_{00}(\gamma)$ be the minimal subset L of $c_{00}(\gamma)$ satisfying the following four properties:

1. For every $\phi \in L$ and every $E \subseteq \gamma$, $E\phi \in L$.
2. For every $\alpha < \gamma$, $\pm e_\alpha^* \in L$.
3. For every $j \in \mathbb{N}$ and every $\phi_1 < \cdots < \phi_{n_j}$ in L, $(1/m_j) \sum_{i=1}^{n_j} \phi_i$ also belongs to L.
4. L is closed under rational convex combinations.

The third property is also described by saying that L is closed in all (m_j^{-1}, n_j)-operations. It is not difficult to see that the norm induced on $c_{00}(\gamma)$ by the set L_γ (i.e., for $x \in c_{00}(\gamma)$, $\|x\| = \sup_{\phi \in L_\gamma} \{\phi x = \langle \phi, x \rangle\}$) is exactly the norm $\| \cdot \|_*$.

REMARK 2.2. Let L'_γ be the minimal subset of $c_{00}(\gamma)$ satisfying 1., 2. and 3. It is not difficult to prove that $L_\gamma = \mathrm{conv}_{\mathbb{Q}}(L'_\gamma)$. This means that L'_γ norms the space $T_\gamma[(m_j^{-1}, n_j)_j]$.

REMARK 2.3. 1. It follows from the minimality of L_γ that each $\phi \in L_\gamma$ is either equal to $\pm e_\alpha$ for some $\alpha < \gamma$ or is of the form $\phi = (1/m_j) \sum_{i=1}^{d} \phi_i$, $d \le n_j$ and $\phi_1 < \cdots < \phi_d$ all in L_γ. Furthermore the set

$$L_{\gamma,j} = \{\phi \in L_\gamma \,:\, \phi = \frac{1}{m_j} \sum_{i=1}^{d} \phi_i\}$$

defines an equivalent norm, denoted by $\| \cdot \|_{*,j}$, on the space $T_\gamma[(m_j^{-1}, n_j)_j]$. The important property of the mixed Tsirelson spaces results from a fine balance of the sequences of norms $(\| \cdot \|_{*,j})_j$. Namely for every block sequence $(x_n)_n$ and for every j there exists a normalized vector y_j in the linear span of $(x_n)_n$ such that $\|y_j\|_{*,j} > 1/4$ and for every $j' \neq j$ $\|y_j\|_{*,j'} < 6/m_{j'}$ if $j' < j$ and $\|y_j\|_{*,j'} < 4/m_j^2$ otherwise.

2.2. **The norming set K_{ω_1}.** The maximal space in our class $\mathfrak{X}_{\omega_1}$ will be defined as the completion of $(c_{00}, \|\cdot\|_\infty)$ under the norm $\|\cdot\|_\infty$ induced by a set of functionals $K_{\omega_1} \subseteq c_{00}(\omega_1)$.

The set K_{ω_1} is the minimal subset of $c_{00}(\omega_1)$ satisfying that:

(1) It contains $(e^*_\gamma)_{\gamma<\omega_1}$, is symmetric (i.e., $\phi \in K$ implies $-\phi \in K$) and is closed under the restriction on intervals of ω_1.

(2) For every $\{\phi_i : i = 1, \dots, n_{2j}\} \subseteq K_{\omega_1}$ with $\operatorname{supp}\phi_1 < \cdots < \operatorname{supp}\phi_{n_{2j}}$, the functional $\phi = (1/m_{2j})\sum_{i=1}^{n_{2j}} \phi_i \in K_{\omega_1}$. We say that ϕ is a result of a (m_{2j}^{-1}, n_{2j})-operation.

(3) For every special sequence $(\phi_1, \dots, \phi_{n_{2j+1}})$ (for a definition, see subsection 2.4), the functional $\phi = (1/m_{2j+1})\sum_{i=1}^{n_{2j+1}} \phi_i$ is in K_{ω_1}. We call ϕ a *special functional* and say that ϕ is a result of a $(m_{2j+1}^{-1}, n_{2j+1})$-operation.

(4) It is rationally convex.

The norm on $c_{00}(\omega_1)$ is defined as $\|x\| = \sup\{\phi(x) = \sum_\alpha \phi(\alpha)\cdot x(\alpha) : \phi \in K_{\omega_1}\}$ and $\mathfrak{X}_{\omega_1}$ is the completion of $(c_{00}(\omega_1), \|\cdot\|)$. Each of the above four properties provides certain features in the space $\mathfrak{X}_{\omega_1}$. The first makes the family $(e_\alpha)_{\alpha<\omega_1}$ a transfinite bimonotone basis of $\mathfrak{X}_{\omega_1}$. The second saturates $\mathfrak{X}_{\omega_1}$ with local unconditional structure. This property will be responsible for the existence of semi-normalized averages in every block sequence of $\mathfrak{X}_{\omega_1}$. The third property saturates $\mathfrak{X}_{\omega_1}$ with conditional structure and will make it impossible for $\mathfrak{X}_{\omega_1}$ to contain any unconditional basic sequence. Finally, the fourth property is a tool for proving properties of the space of operators from an arbitrary subspace X of $\mathfrak{X}_{\omega_1}$ into $\mathfrak{X}_{\omega_1}$. The above definition, with the exception of the missing definition of special sequences, is based on the corresponding definitions from [6] and [7] which in turn are variants of the construction from [12]. By the minimality of K_{ω_1} each $\phi \in K_{\omega_1}$ has one of the following forms:

(i) ϕ is of *type 0* if $\phi = \pm e^*_\alpha$.

(ii) ϕ is of *type I* if $\phi = \pm Ef$ for f a result of one (m_j^{-1}, n_j)-operation and E an interval. In this case we say that the weight $w(\phi)$ of ϕ is m_j.

(iii) ϕ is of *type II* if ϕ is a rational convex combination of type 0 and type I functionals.

An alternative description of the norm is the following: For a given $x \in \mathfrak{X}_{\omega_1}$,

$$\|x\|_* = \max\{\|x\|_\infty, \sup_j \sup \frac{1}{m_{2j}} \sum_{i=1}^{n_{2j}} \|E_i x\|_*,\ E_1 < \cdots < E_{n_{2j}}\} \vee$$

$$\sup\{\frac{1}{m_{2j+1}} \sum_{i=1}^{n_{2j+1}} \phi_i(x) \ :\ \{\phi_i\}_{i=1}^{n_{2j+1}} \text{ is a } n_{2j+1}\text{-special sequence}\}.$$

Remark 2.4. From the definition of the norming set K_{ω_1} it follows easily that $(e_\alpha)_{\alpha<\omega_1}$ is a bimonotone basis of $\mathfrak{X}_{\omega_1}$. Also, it is not difficult to see using *(2)* from the definition of K_{ω_1} that the basis $(e_\alpha)_{\alpha<\omega_1}$ is boundedly complete. Indeed, for $x \in c_{00}(\omega_1)$ and $E_1 < \cdots < E_{n_{2j}}$ intervals of ω_1 *(2)* of the norming set yields that $\|x\| \geq (1/m_{2j})\sum_{i=1}^{n_{2j}} \|E_i x\|$. Also, from the choice of the sequence $(m_j)_j$, $(n_j)_j$, it follows that n_{2j}/m_{2j} increases to infinity. From these observations it follows that the basis $(e_\alpha)_{\alpha<\omega_1}$ is boundedly complete. To prove that the space $\mathfrak{X}_{\omega_1}$ is reflexive we need to show that the basis is shrinking.

Definition 2.5. For $\phi \in K_{\omega_1}$, we say that $m_j \in \mathbb{N}$ is a *weight* of ϕ, or $w(\phi) = m_j$, if ϕ can be obtained as a result of the (m_j^{-1}, n_j)-operation. Notice that $\phi \in K_{\omega_1}$ may have many weights.

The definition of the special sequences will, as in the case [12], depend crucially on certain coding σ_ϱ. The essential difference is that now σ_ϱ is not an injection, a crucial property on which the proofs in [12] rely. Our proofs on the other hand will rely on a "tree-like property" of our coding which we now describe. First we notice that each $2j+1$-special sequence $\Phi = (\phi_1, \phi_2, \dots, \phi_{n_{2j+1}})$ is of the form $\operatorname{supp}\phi_1 < \cdots < \operatorname{supp}\phi_{n_{2j+1}}$ with each ϕ_i of type I. The *tree-like property* is the following: For any pair of $2j+1$-special sequences $\Phi = (\phi_1, \phi_2, \dots, \phi_{n_{2j+1}})$, $\Psi = (\psi_1, \psi_2, \dots, \psi_{n_{2j+1}})$ there exist $1 \le \kappa_{\Phi,\Psi} \le \lambda_{\Phi,\Psi} \le n_{2j+1}$ such that

(i) If $1 \le k < \kappa_{\Phi,\Psi}$ then $\phi_k = \psi_k$ and if $\kappa_{\Phi,\Psi} < k < \lambda_{\Phi,\Psi}$, then $w(\phi_k) = w(\psi_k)$.
(ii) $(\cup_{\kappa_{\Phi,\Psi} < k < \lambda_{\Phi,\Psi}} \operatorname{supp}\phi_k) \cap (\cup_{\kappa_{\Phi,\Psi} < k < \lambda_{\Phi,\Psi}} \operatorname{supp}\psi_k) = \emptyset$.
(iii) $\{w(\phi_k) : \lambda_{\Phi,\Psi} < k < n_{2j+1}\} \cap \{w(\psi_k) : \lambda_{\Phi,\Psi} < k < n_{2j+1}\} = \emptyset$.

Comparing the above tree-like property with the corresponding property from [12], we notice that the new ingredient is the number $\kappa_{\Phi,\Psi}$. Its occurrence is a byproduct of the fact that the coding σ_ϱ is not one-to-one. The property (ii) will however give sufficient control of our special functionals. The coding σ_ϱ is based on the following mapping introduced in [26] (see also [27]).

2.3. **ϱ-functions.** A function $\varrho : [\omega_1]^2 \to \omega$ such that:

1. $\varrho(\alpha,\gamma) \le \max\{\varrho(\alpha,\beta), \varrho(\beta,\gamma)\}$ for all $\alpha < \beta < \gamma < \omega_1$.
2. $\varrho(\alpha,\beta) \le \max\{\varrho(\alpha,\gamma), \varrho(\beta,\gamma)\}$ for all $\alpha < \beta < \gamma < \omega_1$.
3. $\{\alpha < \beta : \varrho(\alpha,\beta) \le n\}$ is finite for all $\beta < \omega_1$ and $n \in \mathbb{N}$.

is called a ϱ-function. The reader is referred to [26] and [27] for full discussion of this notion and constructions of various ϱ-functions. In Section 6 we shall give yet another construction of a ϱ-function with certain universality property.

Let $\varrho : [\omega_1]^2 \to \omega$ be a ϱ-function fixed from now on, and all definitions and facts that follow should be relative to this choice of ϱ.

Definition 2.6. Given a finite set $F \subseteq \omega_1$, let $p_F = p_\varrho(F) = \max_{\alpha,\beta\in F} \varrho(\alpha,\beta)$. For a finite set $F \subseteq \omega_1$ and $p \in \mathbb{N}$, let

$$\overline{F}^p = \{\alpha \le \max F \,:\, \text{there is } \beta \in F \text{ s.t. } \alpha \le \beta \text{ and } \varrho(\alpha,\beta) \le p\}.$$

Notice that by condition 3., $\overline{F}^p$ is a finite set of countable ordinals. We say that F is *p-closed* iff $\overline{F}^p = F$, and that F is ϱ-closed iff it is p_F-closed.

Remark 2.7. 1. Note that $\overline{\cdot}^p$ is a monotone and idempotent operator and so, in particular, every $\overline{F}^p$ is a p-closed set: It is clear that if $F \subseteq G$, then $\overline{F}^p \subseteq \overline{G}^p$. Let us show now that $\overline{\overline{F}^p}^p = \overline{F}^p$. Let $\alpha \in \overline{\overline{F}^p}^p$. This implies that $\varrho(\alpha,\alpha_0) \le p$, for some $\alpha_0 \in \overline{F}^p$, $\alpha \le \alpha_0$. Choose $\alpha_1 \ge \alpha_0$, $\alpha_1 \in F$ such that $\varrho(\alpha_0,\alpha_1) \le p$. Then, $\varrho(\alpha,\alpha_1) \le \max\{\varrho(\alpha,\alpha_0), \varrho(\alpha_0,\alpha_1)\} \le p$.
2. Suppose that $F \subseteq \omega_1$ is finite and suppose that $p \ge p_F$. Then $p_{\overline{F}^p} \le p$. Indeed, let $\alpha < \beta$ such that both belong to $\overline{F}^p$. Let $\alpha' \ge \alpha$, $\beta' \ge \beta$ such that $\alpha, \beta' \in F$ and $\varrho(\alpha,\alpha'), \varrho(\beta,\beta') \le p$. Then we distinguish the following cases:
(a) If $\alpha \le \alpha' \le \beta \le \beta'$, then $\varrho(\alpha,\beta) \le \max\{\varrho(\alpha,\alpha'), \varrho(\alpha',\beta)\} \le \max\{\varrho(\alpha,\alpha'), \varrho(\alpha',\beta'), \varrho(\beta,\beta')\}$ $\le p$.

(b) If $\alpha \le \beta \le \alpha' \le \beta'$, then $\varrho(\alpha,\beta) \le \max\{\varrho(\alpha,\alpha'),\varrho(\beta,\alpha')\} \le \max\{\varrho(\alpha,\alpha'),\varrho(\beta,\beta'),\varrho(\alpha',\beta')\}$ $\le p$.
(c) If $\alpha \le \beta \le \beta' \le \alpha'$, use a similar proof to case (a).

Proposition 2.8. *Let $F, G \subseteq \omega_1$ be two finite sets and $p \ge p_F, p_G$. Then:*

1. *For every ordinal $\alpha \le \omega_1$, $\overline{F\cap\alpha}^p = \overline{F}^p \cap \alpha$ and $\overline{F\cap\alpha}^p$ is an initial part of $\overline{F}^p$. Therefore, if F is p-closed, so is $F\cap\alpha$.*
2. *For every $\alpha \in F\cap G$, we have that $\overline{F\cap(\alpha+1)}^p = \overline{G\cap(\alpha+1)}^p$. Hence, if F and G are in addition p-closed, then $F\cap(\alpha+1) = G\cap(\alpha+1)$.*
3. *$\overline{F\cap G}^p = \overline{F}^p \cap \overline{G}^p$. Therefore, if F and G are p-closed then $F\cap G$ is also p-closed and it is an initial part of both F and G.*

Proof. 1: Since $F\cap\alpha \subseteq F,\alpha$, it follows that $\overline{F\cap\alpha}^p \subseteq \overline{F}^p\cap\alpha$. Now let $\beta\in\overline{F}^p\cap\alpha$. Then there is some $\gamma\in F$, $\gamma \ge be$ such that $\varrho(\beta,\gamma)\le p$. If $\gamma<\alpha$, then we are done. If not, let $\delta = \max F\cap\alpha \in F$ and since $\beta\le\delta<\gamma$ we have that

$$\varrho(\beta,\delta) \le \max\{\varrho(\beta,\gamma),\varrho(\delta,\gamma)\} \le \max\{p,p_F\} = p, \tag{5}$$

the last equality using our assumption that $p\ge p_F$. (5) shows that $\beta\in\overline{F\cap\alpha}^p$. Suppose now that F is p-closed. Then we have just shown that $\overline{F\cap\alpha}^p = \overline{F}^p\cap\alpha = F\cap\alpha$, and we are done.

2: Fix $\alpha\in F\cap G$. Let $\beta\in\overline{F\cap(\alpha+1)}^p = \overline{F}^p\cap(\alpha+1)$. Let $\gamma\in F\cap(\alpha+1)$, $\gamma\ge\beta$ be such that $\varrho(\beta,\gamma)\le p$. Then $\varrho(\beta,\alpha)\le\max\{\varrho(\beta,\gamma),\varrho(\gamma,\alpha)\}\le\max\{p,p_F\}=p$. Since G is p-closed, and $\alpha\in G$, we can conclude that $\beta\in\overline{G\cap(\alpha+1)}^p$. This shows that $\overline{F\cap(\alpha+1)}^p \subseteq \overline{G\cap(\alpha+1)}^p$. The other inclusion follows by symmetry. The last part of 2. follows easily.

3: Let $\alpha = \max F\cap G$. Then by 2., $\overline{F\cap G}^p = \overline{F\cap G\cap(\alpha+1)}^p = \overline{F\cap(\alpha+1)}^p = \overline{F}^p\cap(\alpha+1)$ and $\overline{F\cap G}^p = \overline{G}^p\cap(\alpha+1)$. Combining the above equalities we get $\overline{F\cap G}^p = \overline{F}^p\cap\overline{G}^p\cap(\alpha+1) = \overline{F}^p\cap\overline{G}^p$, the last equality because $\overline{F}^p\cap\overline{G}^p \subseteq F\cap G \subseteq \max(F\cap G)+1 = \alpha+1$. □

2.4. **The σ_ϱ-coding and the special sequences.** We denote by $\mathbb{Q}_s(\omega_1)$ the set of finite sequences $(\phi_1,w_1,p_1,\phi_2,w_2,p_2,\dots,\phi_d,w_d,p_d)$ such that

1. for all $i\le d$, $\phi_i\in c_{00}(\omega_1)$ and $\phi_1<\phi_2<\cdots<\phi_d$,
2. $(w_i)_{i=1}^d$, $(p_i)_{i=1}^d \in \mathbb{N}^d$ are strictly increasing, and
3. $p_i \ge p_{(\cup_{k=1}^i \operatorname{supp}\phi_k)}$ for every $i\le d$.

Let $\mathbb{Q}_s$ be the set of finite sequences $(\phi_1,w_1,p_1,\phi_2,w_2,p_2,\dots,\phi_d,w_d,p_d)$ satisfying 1., and 2. above and in addition for every $i\le d$, $\phi_i\in c_{00}(\mathbb{N})$. Notice that $\mathbb{Q}_s$ is a countable set. Fix a one-to-one function $\sigma:\mathbb{Q}_s\to\{2j:\ j \text{ odd}\}$ such that

$$\sigma(\phi_1,w_1,p_1,\phi_2,w_2,p_2,\dots,\phi_d,w_d,p_d) > \max\{p_d^2,\frac{1}{\varepsilon^2},\max\operatorname{supp}\phi_d\},$$

where $\varepsilon = \min\{|\phi_k(e_\alpha)|:\ \alpha\in\operatorname{supp}\phi_k,\ k=1,\dots,d\}$. Given a finite subset F of ω_1, we denote by $\pi_F:\{1,2,\dots,\#F\}\to F$ the natural order preserving map. Given $\Phi=(\phi_1,w_1,p_1,\phi_2,w_2,p_2,\dots,\phi_d,w_d,p_d)\in\mathbb{Q}_s(\omega_1)$ we set $G_\Phi = \overline{\cup_{i=1}^d \operatorname{supp}\phi_i}^{p_d}$ and then we consider the family

$$\pi_{G_\Phi}(\Phi) = (\pi_G(\phi_1),w_1,p_1,\pi_G(\phi_2),w_2,p_2,\dots,\pi_G(\phi_d),w_d,p_d)\in\mathbb{Q}_s,$$

where

$$\pi_G(\phi_k)(n) = \begin{cases} \phi_k(\pi_{G_\Phi}(n)) & \text{if } n \in G_\Phi \\ 0 & \text{otherwise.} \end{cases}$$

Finally, $\sigma_\varrho : \mathbb{Q}_s(\omega_1) \to \{2j : j \text{ odd}\}$ is defined as $\sigma_\varrho(\Phi) = \sigma(\pi_G(\Phi))$.

A sequence $\Phi = (\phi_1, \dots, \phi_{n_{2j+1}})$ of functionals of K_{ω_1} is said to be a $2j+1$-*special sequence* if:

(1) $\operatorname{supp}\phi_1 < \operatorname{supp}\phi_2 < \cdots < \operatorname{supp}\phi_{n_{2j+1}}$, each ϕ_k is of type I, $w(\phi_k) = m_{2j_k}$ and $w(\phi_1) = m_{2j_1}$ with j_1 even and satisfying $m_{2j_1} > n_{2j+1}^2$.

(2) There exists a strictly increasing sequence $(p_1^\Phi, \dots, p_{n_{2j+1}-1}^\Phi)$ of natural numbers such that for all $1 \le i \le n_{2j+1}-1$ we have that $w(\phi_{i+1}) = m_{\sigma_\varrho(\Phi_i)}$ where $\Phi_i = (\phi_1, w(\phi_1), p_1^\Phi, \phi_2, w(\phi_2), p_2^\Phi, \dots, \phi_i, w(\phi_i), p_i^\Phi)$.

As we have mentioned before, the weight of a type I element of K_{ω_1} is not uniquely determined. However in the case of the elements ϕ_i of a $2j+1$-special sequence Φ, $w(\phi_i)$ will denote the unique weight involved in the definition of the special sequence Φ.

Lemma 2.9 (*Tree-like interference of a pair of special sequences*)**.** *Let $\Phi = (\phi_1, \dots, \phi_{n_{2j+1}})$ and $\Psi = (\psi_1, \dots, \psi_{n_{2j+1}})$ be two $2j+1$-special sequences. Then there are two numbers $0 \le \kappa_{\Phi,\Psi} \le \lambda_{\Phi,\Psi} \le n_{2j+1}$ such that the following conditions hold:*

TP.1 *For all $i \le \lambda_{\Phi,\Psi}$, $w(\phi_i) = w(\psi_i)$ and $p_i^\Phi = p_i^\Psi$.*

TP.2 *For all $i < \kappa_{\Phi,\Psi}$, $\phi_i = \psi_i$.*

TP.3 *For all $\kappa_{\Phi,\Psi} < i < \lambda_{\Phi,\Psi}$*

$$\operatorname{supp}\phi_i \cap \overline{\operatorname{supp}\psi_1 \cup \cdots \cup \operatorname{supp}\psi_{\lambda_{\Phi,\Psi}-1}}^{p_{\lambda_{\Phi,\Psi}-1}} = \emptyset$$
$$\textit{and } \operatorname{supp}\psi_i \cap \overline{\operatorname{supp}\phi_1 \cup \cdots \cup \operatorname{supp}\phi_{\lambda_{\Phi,\Psi}-1}}^{p_{\lambda_{\Phi,\Psi}-1}} = \emptyset.$$

TP.4 $\{w(\phi_i) : \lambda_{\Phi,\Psi} < i \le n_{2j+1}\} \cap \{w(\psi_i) : i \le n_{2j+1}\} = \emptyset$ *and* $\{w(\psi_i) : \lambda_{\Phi,\Psi} < i \le n_{2j+1}\} \cap \{w(\phi_i) : i \le n_{2j+1}\} = \emptyset$.

We refer to the reader to Figure 1 for a description of the conclusion of this Lemma.

Proof. First we observe that for $i \ne l$, $w(\phi_i) \ne w(\phi_l)$. Indeed if $i = 1$ and $l > 1$ then $w(\phi_1) = m_{2j_1}$ with j_1 even while $w(\psi_l) = m_{2j'_l}$ with j'_l odd. If i and l are greater than 1 then $w(\phi_i) = m_{2j_i} \ne m_{2j'_l} = w(\psi_l)$ as consequence of the fact that they code sequences of different lengths $i-1$ and $l-1$ respectively.

Let $\lambda_{\Phi,\Psi}$ be the maximum of all $i \le n_{2j+1}$ such that $w(\phi_i) = w(\psi_i)$ if defined. If not, we set $\lambda_{\Phi,\Psi} = \kappa_{\Phi,\Psi} = 0$. Suppose now that $\lambda_{\Phi,\Psi} > 0$. Define $\kappa_{\Phi,\Psi}$ by

$$\kappa_{\Phi,\Psi} = \min\{i < \lambda_{\Phi,\Psi} : \phi_i \ne \psi_i\},$$

if defined and $\kappa_{\Phi,\Psi} = 0$ if not. In this last case it is trivial to check our requirements. So assume that $\kappa_{\Phi,\Psi} > 0$. (TP.2) and (TP.4) follows easily from the properties of the coding σ_ϱ. We show (TP.3). Let

$$G = \overline{\bigcup_{i=1}^{\lambda_{\Phi,\Psi}-1} \operatorname{supp}\phi_i}^{p_{\lambda_{\Phi,\Psi}-1}} \quad \text{and } G' = \overline{\bigcup_{i=1}^{\lambda_{\Phi,\Psi}-1} \operatorname{supp}\psi_i}^{p_{\lambda_{\Phi,\Psi}-1}}.$$

And let $\pi_G : G \to \{1, \dots, \#G\}$ and $\pi'_G : G' \to \{1, \dots, \#G\}$ be the unique order-preserving bijections.

Claim. *(a)* $\#G = \#G'$.
(b) $\pi_G|(G \cap G') = \pi_{G'}|(G \cap G')$ *and* $(G \cap G')\phi_{\kappa_{\Phi,\Psi}} = (G \cap G')\psi_{\kappa_{\Phi,\Psi}}$.
(c) $\max(G \cap G') < \min\{\max\operatorname{supp}\phi_{\kappa_{\Phi,\Psi}}, \max\operatorname{supp}\psi_{\kappa_{\Phi,\Psi}}\}$.

Proof of Claim: (a): Notice that

$$\#G = \max\operatorname{supp}\pi_G(\phi_{\lambda_{\Phi,\Psi}-1}) \text{ and } \#G' = \max\operatorname{supp}\pi_{G'}(\psi_{\lambda_{\Phi,\Psi}-1}). \tag{6}$$

Since $\sigma_\varrho((\phi_i, w(\phi_i), p_i)_{i=1}^{\lambda_{\Phi,\Psi}-1}) = \sigma_\varrho((\psi_i, w(\psi_i), p_i)_{i=1}^{\lambda_{\Phi,\Psi}-1})$, then $\pi_G(\phi_{\lambda_{\Phi,\Psi}-1}) = \pi_{G'}(\psi_{\lambda_{\Phi,\Psi}-1})$ and hence $\#G = \#G'$, as desired. (b): It follows from the properties of ϱ that $\pi_G|(G \cap G') = \pi_{G'}|(G \cap G')$ (see Remark 2.7 and Proposition 2.8). Fix now $\alpha \in G \cap G'$. Since $\pi_G(\alpha) = \pi_{G'}(\alpha)$ we have that

$$\phi_{\kappa_{\Phi,\Psi}}(e_\alpha) = \psi_{\kappa_{\Phi,\Psi}}(e_{\pi_G(\pi_{G'}^{-1}\alpha)}) = \psi_{\kappa_{\Phi,\Psi}}(e_\alpha), \tag{7}$$

as desired. (c): Suppose not. W.l.o.g. assume that $\max G \cap G' \geq \max\operatorname{supp}\phi_{\kappa_{\Phi,\Psi}}$. (b) yields that

$$\phi_{\kappa_{\Phi,\Psi}} = (G \cap G')\phi_{\kappa_{\Phi,\Psi}} = (G \cap G')\psi_{\kappa_{\Phi,\Psi}}, \tag{8}$$

and since $\#\operatorname{supp}\phi_{\kappa_{\Phi,\Psi}} = \#\operatorname{supp}\psi_{\kappa_{\Phi,\Psi}}$ we obtain that $\phi_{\kappa_{\Phi,\Psi}} = \psi_{\kappa_{\Phi,\Psi}}$, a contradiction. $\square$

To complete the proof choose $\kappa_{\Phi,\Psi} < i < \lambda_{\Phi,\Psi}$. Then the previous Claim yields that $\operatorname{supp}\phi_i \subseteq G \setminus (G \cap G')$ and hence $\operatorname{supp}\phi_i \cap G' = \emptyset$. $\square$

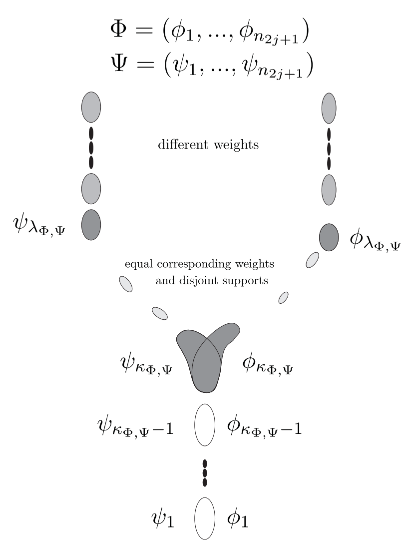

FIGURE 1. The tree-like interference between a pair of special sequences.

2.5. **Tree-analysis of functionals.** Computing the norm of a vector x from $\mathfrak{X}_{\omega_1}$ is typically not an easy task. From the definition of the norming set K_{ω_1} one observes that each $\phi \in K_{\omega_1}$ is constructed from the basic functionals e_α^* after finitely many steps where at each step one applies some (m_j^{-1}, n_j)-operation, or one takes some convex combination. The tree-analysis defined below describes this procedure and it will be a very useful tool in estimations of the norm of certain vectors of $\mathfrak{X}_{\omega_1}$.

Definition 2.10. Let ϕ be a functional of the norming set K_{ω_1}. A *tree-analysis* of ϕ is a mapping $\mathcal{F} : \mathcal{T} \to K_{\omega_1}$, $t \mapsto \mathcal{F}(t) = \phi_t$ such that the following conditions are satisfied:

1. $\mathcal{T} = (\mathcal{T}, \prec)$ is a finite tree with a unique root $\emptyset \in \mathcal{T}$, and $\phi_\emptyset = \phi$.
2. If $t \in \mathcal{T}$ is a maximal node of $\mathcal{T}$, then $\phi_t = \pm e_\alpha^*$, for some $\alpha < \omega_1$. We say in this case that $\phi_t = \pm e_\alpha^*$ has *type 0.*
3. If $t \in \mathcal{T}$ is not a maximal node, and denoting by S_t the set of immediate successors of t, S_t satisfies exactly one of the following two:
   (3.a) There is a unique ordering of $S_t = \{s_1 <_t \cdots <_t s_d\}$ defined by $\phi_{s_1} < \cdots < \phi_{s_d}$, there exists an integer $j \in \mathbb{N}$ such that $d \leq n_j$ and $\phi_t = (1/m_j) \sum_{i=1}^d \phi_{s_i}$.
   (3.b) For every $s \in S_t$, ϕ_s is either of type 0 or I, and there is a sub-convex family $\{r_u\}_{u \in S_t}$ of positive rational numbers such that $\phi_t = \sum_{s \in S_t} r_s \phi_s$.
4. For every $s \preceq t$, $\operatorname{ran} \phi_t \subseteq \operatorname{ran} \phi_s$.

For a given $\phi \in K_{\omega_1}$, whenever we write $w(\phi)$ we implicitly assume that ϕ is of type I. In many cases we will use the explicit notation $(\phi_t)_{t \in \mathcal{T}}$ to denote a tree-analysis.

Remark 2.11. 1. The minimality of K_{ω_1} easily yields that for any functional $f \in K_{\omega_1}$ there is a tree $(f_t)_{t \in \mathcal{T}}$ satisfying conditions 1-3. Such a tree $(f_t)_{t \in \mathcal{T}}$ for f can be refined to a tree-analysis of f. The proof goes as follows: Given a tree-analysis $(f_t)_{t \in \mathcal{T}}$ of f we show by downwards induction over $\mathcal{T}$ that every f_t has a tree-analysis as desired. The only non trivial case is when f_t is of type II, $f_t = \sum_{s \in S_t} r_s f_s$. Let $E = \operatorname{ran} f_t$, and let $f_s' = f_s | E$ for $s \in S_t$. Then, $f_t = f_t | E = \left(\sum_{s \in S_t} r_s f_s\right) | E = \sum_{s \in S_t} r_s f_s'$. Since $\operatorname{ran} f_s' \subseteq E = \operatorname{ran} f_t$, the inductive hypothesis finishes the proof.
2. Observe that the subset of K_{ω_1} consisting on functionals of type 0 and I is also a norming set for the space: Given a finitely supported vector x, and $\phi = \sum_i r_i \phi_i$ of type II with ϕ_i of type 0 or I, $|\langle \phi, x \rangle| = |\sum_i r_i \langle \phi_i, x \rangle| \leq \max_i |\langle \phi_i, x \rangle|$.
3. Observe that a given $\phi \in K_{\omega_1}$ may have many trees as well as weights.

3. $\mathfrak{X}_{\omega_1}$ has no unconditional basic sequences

Definition 3.1. A pair (x, ϕ) with $x \in \mathfrak{X}_{\omega_1}$ and $\phi \in K_{\omega_1}$ is said to be a (C, j)*-exact pair* if *(a)* $\|x\| \leq C$, $w(\phi) = m_j$ and $\phi(x) = 1$, and *(b)* for every $\psi \in K_{\omega_1}$ of type I and $w(\psi) = m_i$, $i \neq j$ we have:

$$|\psi(x)| \leq \begin{cases} \frac{2C}{m_i} & \text{if } i < j \\ \frac{C}{m_j^2} & \text{if } i > j. \end{cases} \tag{9}$$

(C, j)-exact pairs are one of the basic ingredients for the study of mixed Tsirelson spaces as well as of hereditarily indecomposable spaces built on a frame of a mixed Tsirelson space. The next proposition ensures their existence everywhere.

Proposition 3.2. *Let $(x_n)_n$ be a block sequence in $\mathfrak{X}_{\omega_1}$. Then for each $j \in \mathbb{N}$ there exists (x, ϕ) such that $x \in \langle x_n \rangle_n$, $\phi \in K_{\omega_1}$ and (x, ϕ) is a $(6, j)$-exact pair.*

The existence of $(6, j)$-exact pairs it is proved by a similar argument to that for the Gowers-Maurey space [12]. It is primarily based on the unconditional part of the definition of K_{ω_1} (i.e., property 2.). A simple example of a $(6, 2j)$-exact pair is the pair (x, ϕ) where $x = (m_{2j}/n_{2j}) \sum_{\alpha \in F} e_\alpha$, $\phi = (1/m_{2j}) \sum_{\alpha \in F} e^*_\alpha$ and $\#F = n_{2j}$.

Definition 3.3. Let $j \in \mathbb{N}$. A sequence $(x_1, \phi_1, \ldots, x_{n_{2j+1}}, \phi_{n_{2j+1}})$ is said to be a $(1, j)$-*dependent sequence* if:

DS.1 $\operatorname{supp} x_1 \cup \operatorname{supp} \phi_1 < \cdots < \operatorname{supp} x_{n_{2j+1}} \cup \operatorname{supp} \phi_{n_{2j+1}}$.
DS.2 The sequence $\Phi = (\phi_1, \ldots, \phi_{n_{2j+1}})$ is a $2j + 1$-special sequence.
DS.3 (x_i, ϕ_i) is a $(6, 2j_i)$-exact pair with $\#\operatorname{supp} x_i \leq m_{2j_{i+1}}/n^2_{2j+1}$ for every $1 \leq i \leq n_{2j+1}$.
DS.4 For every $(2j + 1)$-special sequence $\Psi = (\psi_1, \ldots, \psi_{n_{2j+1}})$ we have that

$$\bigcup_{\kappa_{\Phi,\Psi} < i < \lambda_{\Phi,\Psi}} \operatorname{supp} x_i \cap \bigcup_{\kappa_{\Phi,\Psi} < i < \lambda_{\Phi,\Psi}} \operatorname{supp} \psi_i = \emptyset. \tag{10}$$

Proposition 3.4. *For every $(y_n)_n$, a block sequence of $\mathfrak{X}_{\omega_1}$, and every $j \in \mathbb{N}$ there exists $(1, j)$-dependent sequence $(x_1, \phi_1, \ldots, x_{n_{2j+1}}, \phi_{n_{2j+1}})$ such that $x_i \in \langle y_n \rangle_n$ for every $i = 1, \ldots, n_{2j+1}$.*

PROOF. Let $(y_n)_n$ and j be given. We inductively produce $\{(x_i, \phi_i)\}_{i=1}^{n_{2j+1}}$ as follows. For $i = 1$ we choose a $(6, 2j_1)$-exact pair (x_1, ϕ_1) such that $m_{2j_1} > m^2_{2j+1}$, j_1 even (see the definition of special sequences) and $x_1 \in \langle y_n \rangle_n$. Assume that $\{(x_l, \phi_l)\}_{l=1}^{i-1}$ has been chosen such that there exists $(p_l)_{l=1}^{i-2}$ satisfying
(a) $\operatorname{supp} x_1 \cup \operatorname{supp} \phi_1 < \cdots < \operatorname{supp} x_{i-1} \cup \operatorname{supp} \phi_{i-1}$, each $x_l \in \langle y_n \rangle_n$ and (x_l, ϕ_l) is a $(6, 2j_l)$-exact pair.
(b) For $1 < l \leq i - 1$, $w(\phi_l) = \sigma_\varrho(\phi_1, w(\phi_1), p_1, \ldots, \phi_{l-1}, w(\phi_{l-1}), p_{l-1})$.
(c) For $1 \leq l < i - 1$, $p_l \geq \max\{p_{l-1}, p_{F_l}\}$, where $F_l = \bigcup_{k=1}^{l} \operatorname{supp} \phi_k \cup \operatorname{supp} x_k$.

To define (x_i, ϕ_i) we choose $p_{i-1} \geq \max\{p_{i-2}, p_{F_{i-1}}, n^2_{2j+1} \cdot \#\operatorname{supp} x_i\}$ and we set

$$2j_i = \sigma_\varrho(\phi_1, w(\phi_1), p_1, \ldots, \phi_{i-1}, w(\phi_{i-1}), p_{i-1}).$$

Choose a $(6, 2j_i)$-exact pair (x_i, ϕ_i) such that $x_i \in \langle y_n \rangle_n$ and $\operatorname{supp} x_{i-1} \cup \operatorname{supp} \phi_{i-1} < \operatorname{supp} x_i \cup \operatorname{supp} \phi_i$. This completes the inductive construction. (*DS.1*)-(*DS.3*) easily holds, while (*DS.4*) follows from *(c)* and (3) of Lemma 2.9. □

REMARK 3.5. Suppose that $(y_n)_n$ and $(z_n)_n$ are block sequences such that $\sup_n \max \operatorname{supp} y_n = \sup_n \max \operatorname{supp} z_n$. Then for every $j \in \mathbb{N}$ there is a $(1, j)$-dependent sequence $(x_1, \phi_1, \ldots, x_{n_{2j+1}}, \phi_{n_{2j+1}})$ with the property that $x_{2i-1} \in \langle y_n \rangle_n$ and $x_{2i} \in \langle z_n \rangle_n$ for every $i = 1, \ldots, n_{2j+1}/2$.

Lemma 3.6. *Fix a $(1, j)$-dependent sequence $(x_1, \phi_1, \ldots, x_{n_{2j+1}}, \phi_{n_{2j+1}})$, and a sequence of scalars $(\lambda_i)_{i=1}^{n_{2j+1}}$ such that $\max_i |\lambda_i| \leq 1$. Suppose that for every $\psi \in K_{\omega_1}$ such that $w(\psi) =$*

m_{2j+1}, and every interval of integers $E \subseteq [1, n_{2j+1}]$ it holds that

$$|\psi(\sum_{i\in E} \lambda_i x_i)| \leq 12(1 + \frac{\#E}{n^2_{2j+1}}). \tag{11}$$

Then,

$$\|\frac{1}{n_{2j+1}} \sum_{i=1}^{n_{2j+1}} \lambda_i x_i\| \leq \frac{1}{m^2_{2j+1}}. \tag{12}$$

We postpone the proof of this lemma to the end of Subsection 4.3, since involves non trivial estimates.

Proposition 3.7. *If $(x_1, \phi_1, \dots, x_{n_{2j+1}}, \phi_{n_{2j+1}})$ is a $(1, j)$-dependent sequence, then*

$$\|\frac{1}{n_{2j+1}} \sum_{i=1}^{n_{2j+1}} x_i\| \geq \frac{1}{m_{2j+1}} \text{ and } \|\frac{1}{n_{2j+1}} \sum_{i=1}^{n_{2j+1}} (-1)^{i+1} x_i\| \leq \frac{1}{m^2_{2j+1}}. \tag{13}$$

Proof. The first estimate is clear since the functional $\phi = (1/m_{2j+1}) \sum_{i=1}^{n_{2j+1}} \phi_i \in K_{\omega_1}$ and $\phi((1/n_{2j+1}) \sum_{i=1}^{n_{2j+1}} x_i) = 1/m_{2j+1}$. For the second, we use Lemma 3.6 applied to the sequence of scalars $((-1)^{i+1})_i$, and the desired estimate will follow from (12). Fix $\psi \in K_{\omega_1}$ with $w(\psi) = m_{2j+1}$, and an interval $E \subseteq [1, n_{2j+1}]$. Set $\Psi = (\psi_1, \dots, \psi_{n_{2j+1}})$ and $x = \sum_{i\in E} (-1)^{i+1} x_i$, where $\psi = (1/m_{2j+1}) \sum_{i\in E} \psi_i$. Notice that

$$|\psi(x)| = |\frac{1}{m_{2j+1}} \sum_{i=1}^{\kappa_{\Phi,\Psi}-1} \phi_i(x) + \frac{1}{m_{2j+1}} \sum_{i=\kappa_{\Phi,\Psi}}^{n_{2j+1}} \psi_i(x)| \leq \frac{1}{m_{2j+1}} + |\frac{1}{m_{2j+1}} \sum_{i=\kappa_{\Phi,\Psi}}^{n_{2j+1}} \psi_i(x)|. \tag{14}$$

We shall show that the following hold

(a) $|\psi_{\kappa_{\Phi,\Psi}}(\sum_{i\in E} (-1)^{i+1} x_i)| \leq 1 + 12(\#E - 1)/n^2_{2j+1}$,
(b) $|\psi_{\lambda_{\Phi,\Psi}}(\sum_{i\in E} (-1)^{i+1} x_i)| \leq 1 + 12(\#E - 1)/n^2_{2j+1}$, and
(c) $|(\sum_{l>\kappa_{\Phi,\Psi}, l\neq\lambda_{\Phi,\Psi}} \psi_l)(x_i)| \leq 12/n_{2j+1}$ for every $1 \leq i \leq n_{2j+1}$.

Let us show first *(a)*. Let $2j_i$ be such that $w(\phi_i) = m_{2j_i}$. Notice that for $i \neq \kappa_{\Phi,\Psi}$ we have that

$$|\psi_{\kappa_{\Phi,\Psi}}(x_i)| \leq \begin{cases} \frac{12}{w(\psi_{\kappa_{\Phi,\Psi}})} & \text{if } i > \kappa_{\Phi,\Psi} \\ \frac{6}{m^2_{2j_i}} & \text{if } i < \kappa_{\Phi,\Psi}. \end{cases} \tag{15}$$

By the properties of the sequences $(m_l)_l$, $(n_l)_l$ and the fact that $n^2_{2j+1} < w(\psi_{\kappa_{\Phi,\Psi}})$, m_{2j_i}, (15) yields that $|\psi_{\kappa_{\Phi,\Psi}}(x_i)| \leq 12/n^2_{2j+1}$ for $i \neq \kappa_{\Phi,\Psi}$. Hence

$$|\psi_{\kappa_{\Phi,\Psi}}(\sum_{i\in E} x_i)| \leq |\psi_{\kappa_{\Phi,\Psi}}(x_{\kappa_{\Phi,\Psi}})| + |\psi_{\kappa_{\Phi,\Psi}}(\sum_{i\in E,\, i\neq\kappa_{\Phi,\Psi}} x_i)| \leq 1 + \frac{12(\#E - 1)}{n^2_{2j+1}}. \tag{16}$$

(b) has a similar proof to that of *(a)*. We check now *(c)*. Fix $l > \kappa_{\Phi,\Psi}$, $l \neq \lambda_{\Phi,\Psi}$. Suppose that $l > \lambda_{\Phi,\Psi}$. Since $w(\psi_l) \neq w(\phi_i)$ for all $i \leq n_{2j+1}$, we obtain that $|\psi_l(x_i)| \leq 12/n^2_{2j+1}$. Now suppose that $\kappa_{\Phi,\Psi} < l < \lambda_{\Phi,\Psi}$. By *(DS.4)* we have that $\psi_l(x_i) = 0$ for every $\kappa_{\Phi,\Psi} < i < \lambda_{\Phi,\Psi}$. And for $i \notin (\kappa_{\Phi,\Psi}, \lambda_{\Phi,\Psi})$, using the fact that $w(\psi_l) \neq w(\phi_i)$, we can conclude that $|\psi_l(x_i)| \leq 12/n^2_{2j+1}$. Hence, $(\sum_{l>\kappa_{\Phi,\Psi}, l\neq\lambda_{\Phi,\Psi}} \psi_l)(x_i) \leq 12/n_{2j+1}$ for every $1 \leq i \leq n_{2j+1}$, as desired.

Combining *(a)*, *(b)* and *(c)* we obtain that

$$|\frac{1}{m_{2j+1}}\sum_{i=\kappa_{\Phi,\Psi}}^{n_{2j+1}}\psi_i(x)| \leq 1+\frac{\#E}{n_{2j+1}^2}. \tag{17}$$

From (14) and (17) we conclude that $|\psi(x)| \leq 12(1+\#E/n_{2j+1}^2)$, as desired. □

Proposition 3.8. *The closed linear span of a block sequence of $\mathfrak{X}_{\omega_1}$ is hereditarily indecomposable.*

Proof. Fix a block sequence $(y_n)_n$ of $\mathfrak{X}_{\omega_1}$, two block subsequences $(z_n)_n$ and $(w_n)_n$ of $(y_n)_n$ and $\varepsilon > 0$. Let j be large enough such that $m_{2j+1}\varepsilon > 1$. By Proposition 3.4 we can choose a $(1, j)$-dependent sequence $(x_1, \phi_1, \dots, x_{n_{2j+1}}, \phi_{n_{2j+1}})$ such that $x_{2i-1} \in \langle z_n\rangle_n$, and $x_{2i} \in \langle w_n\rangle_n$. Set $z = (1/n_{2j+1})\sum_{i=1,i \text{ odd}}^{n_{2j+1}} x_i$ and $w = (1/n_{2j+1})\sum_{i=1,i \text{ even}}^{n_{2j+1}} x_i$. Notice that $z \in \langle z_n\rangle_n$ and $w \in \langle w_n\rangle_n$. By Proposition 3.6, we know that $\|z + w\| \geq 1/m_{2j+1}$ and $\|z - w\| \leq 1/m_{2j+1}^2$. Hence $\|z - w\| \leq \varepsilon\|z + w\|$. □

Corollary 3.9. (a) *The distance between the unit spheres of every two normalized block sequences (x_n) and (y_n) in $\mathfrak{X}_{\omega_1}$ such that* $\sup_n \max \operatorname{supp} x_n = \sup_n \max \operatorname{supp} y_n$ *is 0.*
(b) *There is no unconditional basic sequence in $\mathfrak{X}_{\omega_1}$.*
(c) *Every infinite dimensional closed subspace of $\mathfrak{X}_{\omega_1}$ contains an hereditarily indecomposable subspace.*
(d) *The distance between the unit spheres of two nonseparable subspaces of $\mathfrak{X}_{\omega_1}$ is equal to 0.*

Proof. (b): follows from Proposition 3.8 and 4. of Proposition 1.3. (c): This result follows from (b) and Gowers' dichotomy. Moreover, every subspace of $\mathfrak{X}_{\omega_1}$ isomorphic to the closed linear span of a block sequence with respect to the basis $(e_\alpha)_{\alpha<\omega_1}$ is hereditarily indecomposable. (d): Fix two nonseparable closed subspaces X and Y of $\mathfrak{X}_{\omega_1}$. Now we can find a sequence $(z_n)_n$ of normalized vectors such that for every n (a) $z_{2n-1} \in X$, $z_{2n} \in Y$ and (b) $\operatorname{supp} z_n < \operatorname{supp} z_{n+1}$. Notice that the supports $\operatorname{supp} z_n$ are not necessarily finite. Now approximating $(z_n)_n$ by a normalized block sequence $(w_n)_n$ as close as needed we obtain the desired result. □

4. Basic estimations and further properties of $\mathfrak{X}_{\omega_1}$

In this section we introduce some of the standard tools of this area (see [22], [12],[7], [4]) which will be quite useful in our analysis of the space $\mathfrak{X}_{\omega_1}$. We also obtain that the space $\mathfrak{X}_{\omega_1}$ is reflexive.

4.1. **Rapidly Increasing Sequences. The basic inequality.**

Definition 4.1. *(Rapidly Increasing Sequences (RIS))* Let $C, \varepsilon > 0$. A block sequence $(x_k)_k$ of X is called a (C, ε)- *rapidly increasing sequence* ((C, ε)-RIS in short) iff there is an increasing sequence $(j_k)_k$ of integers such that for all k,

1. $\|x_k\| \leq C$
2. $|\operatorname{supp} x_k| \leq m_{j_{k+1}}\varepsilon$ and
3. For all type I functionals ϕ of K with $w(\phi) < m_{j_k}$, $|\phi(x_k)| \leq C/w(\phi)$.

Remark 4.2. *1.* Notice that given $\varepsilon' < \varepsilon$, every (C, ε)-RIS has a subsequence which is (C, ε')-RIS. Notice also that for every strictly increasing sequence $\{\alpha_n\}_n$, and every $\varepsilon > 0$, $(e_{\alpha_n})_n$ is a $(1, \varepsilon)$-RIS. *2.* For every $(1, j)$-dependent sequence $(x_1, \phi_1, \dots, x_{n_{2j+1}}, \phi_{n_{2j+1}})$ the corresponding block sequence $(x_1, \dots, x_{n_{2j+1}})$ is a $(12, 1/n_{2j+1}^2)$-RIS.

A primary reason for the usefulness of the notion of RIS is that one has good estimates of upper bounds on $|\langle f, x\rangle|$, for $f \in K_{\omega_1}$ and x averages of an RIS.

Notation. In the sequel we shall denote by W the minimal subset of $c_{00}(\mathbb{N})$ which contains $\{e_n^*\}_{n\in\mathbb{N}}$, is symmetric, and is closed in rational convex combinations, closed in restriction to intervals, and closed for the $(m_j^{-1}, 4n_j)$-operations.

Remark 4.3. By minimality of W, every element f of W has a tree-analysis $(f_t)_{t\in\mathcal{T}}$. Using induction over the tree-analysis, it is not difficult to show that every $f \in K$ is the convex combination $f = \sum_i r_i f_i$, with every f_i in the norming set of $T[(m_j^{-1}, 4n_j)_j]$ and in the case that f is of type I, then each f_i can be chosen such that $w(f_i) = w(f)$. Hence, W norms the mixed Tsirelson space $T[(m_j^{-1}, 4n_j)_j]$.

The following Lemma gives a very useful tool for reducing for a given $f \in K_{\omega_1}$ and a RIS $(x_k)_k$, upper bound estimates of $|\langle f, \sum_k b_k x_k\rangle|$ to upper bounds of $|\langle g, \sum_k |b_k| e_k\rangle|$ where g is a functional of the auxiliary space $T[(m_j^{-1}, 4n_j)_j]$ and $(e_k)_k$ is its basis.

Lemma 4.4 (*Basic Inequality for RIS*)**.** *Let $(x_n)_n$ be a (C, ε)-RIS sequence and fix $(b_k)_k \in c_{00}(\mathbb{N})$. Suppose that $j_0 \in \mathbb{N}$ is such that for all $f \in K_{\omega_1}$ with weight $w(f) = m_{j_0}$ and all intervals E,*

$$\left| f(\sum_{k\in E} b_k x_k) \right| \leq C \left(\max_{k\in E} |b_k| + \varepsilon \sum_{k\in E} |b_k| \right). \tag{18}$$

(We say in this case that $(x_n)_n$ makes j_0 negligible *for $(b_k)_k$.) Then for every $f \in K_{\omega_1}$ of type I there exists $g_1, g_2 \in c_{00}(\mathbb{N})$ such that*

$$|f(\sum b_k x_k)| \leq C(g_1 + g_2)(\sum |b_k| e_k),$$

where $g_1 = h_1$ or $g_1 = e_t^* + h_1$, $t \notin \operatorname{supp} h_1$, and $h_1 \in W$ is such that $h_1 \in \operatorname{conv}_{\mathbb{Q}}\{h \in W : w(h) = w(f)\}$ and with m_{j_0} not appearing as a weight of a node of a tree-analysis of h_1, and $\|g_2\|_\infty \leq \varepsilon$.

We postpone the proof of this result until Subsection 8.2.

Remark 4.5. Notice that any finite (C, ε)-RIS sequence $(x_k)_k$ is going to be j_0-negligible for large j_0.

4.2. **Estimates on the basis.**

Proposition 4.6. *Fix a functional f of type I, either in W or in K_{ω_1}, $j \in \mathbb{N}$ and $l \in [n_j/m_j, n_j]$. Then for every set $\#F = l$*

$$|f(\frac{1}{l}\sum_{\alpha\in F} e_\alpha)| \leq \begin{cases} \frac{2}{w(f)m_j} & \text{if } w(f) < m_j \\ \frac{1}{w(f)} & \text{if } w(f) \geq m_j. \end{cases} \tag{19}$$

If the tree-analysis of f does not contain nodes with weight m_j, then

$$|f(\frac{1}{l}\sum_{\alpha\in F} e_\alpha)| \leq \frac{2}{m_j^3}, \tag{20}$$

where in each case we interpret $(e_\alpha)_{\alpha\in F}$ in the obvious way.

PROOF. Fix $f \in W$ of type I. By Remark 4.3 we can assume that f belongs to the norming set of $T[(m_j^{-1}, 4n_j)_j]$, i.e., f admits a tree-analysis with no convex combinations. The result is proved in the same manner as Lemma 4.2 of [6].

The result for $f \in K_{\omega_1}$ follows easily from the following. Let us denote by $\|\cdot\|_l$ the norm of the natural extension of $T[(m_j^{-1}, 4n_j)_j]$ to ω_1. It is clear that for this norm the natural Hamel basis $(e_\alpha)_{\alpha<\omega_1}$ of $c_{00}(\omega_1)$ is 1-subsymmetric, and also that $\|\cdot\|_l$ dominates the norm $\|\cdot\|_{\mathfrak{X}_{\omega_1}}$. □

4.3. **Consequences of the basic inequality.** We start this subsection with the following estimates on averages of RIS.

Proposition 4.7. *Let $(x_k)_k$ be a (C,ε)-RIS for $\varepsilon \leq 1/n_j$, $l \in [n_j/m_j, n_j]$ and let $f \in K$ of type I. Then,*

$$|f(\frac{1}{l}\sum_{k=1}^{l} x_k)| \leq \begin{cases} \frac{3C}{w(f)m_j} & \textit{if } w(f) < m_j \\ \frac{C}{w(f)} + \frac{2C}{n_j} & \textit{if } w(f) \geq m_j. \end{cases} \tag{21}$$

Consequently, if $(x_k)_{k=1}^l$ is a normalized (C,ε)-RIS with $\varepsilon \leq 1/n_{2j}$ and $l \in [n_{2j}/m_{2j}, n_{2j}]$, then

$$\frac{1}{m_{2j}} \leq \|\frac{1}{l}\sum_{k=1}^{l} x_k\| \leq \frac{2C}{m_{2j}}. \tag{22}$$

PROOF. This follows from the basic inequality and the estimates on the basis of $T[(m_j^{-1}, 4n_j)_j]$ given in Proposition 4.6. For the last consequence, notice that if for every $k \leq l$ we consider x_k^* in K such that $x_k^* x_k = 1$ and $\operatorname{ran} x_k^* \subseteq \operatorname{ran} x_k$, then $x^* = (1/m_{2j})\sum_{k=1}^{l} x_k^*$ belongs to K, and $x^*((1/n_{2j})\sum_{k=1}^{l} x_k) = 1/m_{2j}$. □

Definition 4.8. Let $C > 0$ and $k \in \mathbb{N}$. A normalized vector y is called a $C - \ell_1^k$-average iff there is a finite block sequence $(x_1, \dots, x_n)$ such that $y = (x_1 + \cdots + x_k)/k$ and $\|x_i\| \leq C$.

Observe that since K_{ω_1} is closed under the (m_{2j}^{-1}, n_{2j})-operation, for every normalized block sequences $(y_n)_n$ and every k, there are $z_1 < \cdots < z_k$ in $\langle y_n\rangle_n$ such that $(z_1 + \cdots + z_k)/k$ is a $2 - \ell_1^k$-average (for a detailed proof see for example [6]).

Proposition 4.9. *Suppose that y is a $C - \ell_1^k$-average and suppose that $E_1 < \cdots < E_n$ are intervals with $n < k$. Then, $\sum_{i=1}^n \|E_i y\| \leq C(1 + 2n/k)$. As a consequence, if y is a $C - \ell_1^{n_j}$-average and $\phi \in K$ is with $w(\phi) < m_j$, then $|\phi(y)| \leq 3C/2w(\phi)$.*

In particular, for $2 - \ell_1^{n_j}$-averages we get that $|\phi(y)| \leq 3/w(\phi)$ if $w(\phi) < m_j$.

PROOF. See [22] or [12]. □

REMARK 4.10. Suppose that $(x_k)_k$ is such that there is a strictly increasing sequence $(j_k)_k$ and $\varepsilon > 0$ such that for all k, (a) x_k is a $2 - \ell_1^{n_{j_k}}$-average and (b) $\#\operatorname{supp} x_k < \varepsilon m_{j_{k+1}}$. Then Proposition 4.9 shows that $(x_k)_k$ is a $(3,\varepsilon)$-RIS. In this case we will say that $(x_k)_k$ is a $(3,\varepsilon)$-RIS of ℓ_1 averages. These remarks yield the following.

Proposition 4.11. *Any block sequence in $\mathfrak{X}_{\omega_1}$ has a further normalized block subsequence which is a $(3,\varepsilon)$-RIS.* $\square$

Proposition 4.12. *Let $(x_n)_n$ be a block sequence in $\mathfrak{X}_{\omega_1}$. Then for each $j \in \mathbb{N}$ there exists a $(6,2j)$-exact pair (x,ϕ) such that $x \in \langle x_n \rangle_n$.*

Proof. Fix a block sequence $(x_n)_n$ of $\mathfrak{X}_{\omega_1}$ and an integer j. By the previous proposition we can find a normalized $(3,1/n_{2j})$-RIS $(y_n)_n$ in $\langle x_n \rangle_n$. For each $1 \le i \le n_{2j}$ choose $\phi_i \in K_{\omega_1}$ such that $\phi_i(y_i) = 1$, and $\phi_i < \phi_{i+1}$. Set $\phi = (1/m_{2j}) \sum_{i=1}^{n_{2j}} \phi_i \in K_{\omega_1}$, and $x = (m_{2j}/n_{2j}) \sum_{i=1}^{n_{2j}} y_i$. Then $\phi(x) = 1$ and estimates in Proposition 4.7 yield

$$|f(x)| \le \begin{cases} \frac{9}{w(f)} & \text{if } w(f) < m_{2j} \\ \frac{3m_{2j}}{w(f)} + \frac{6m_{2j}}{n_{2j}} & \text{if } w(f) \ge m_{2j}, \end{cases} \tag{23}$$

and $\|x\| \le 6$. Hence (x,ϕ) is a $(6,2j)$-exact pair. $\square$

To finish this subsection we show Lemma 3.6:

Proof. (Of Lemma 3.6.) Fix a $(1,j)$ dependent sequence $(x_1,\phi_1,\dots,x_{n_{2j+1}},\phi_{n_{2j+1}})$ and a sequence $(\lambda_i)_{i=1}^{n_{2j+1}}$ with $\max_i |\lambda_i| \le 1$ such that for every ψ with weight m_{2j+1}, and every interval $E \subseteq [1, n_{2j+1}]$,

$$|\psi(\sum_{i\in E} \lambda_i x_i)| \le 12(1 + \frac{\#E}{n_{2j+1}^2}). \tag{24}$$

Since $(x_i)_i$ is a $(12, 1/n_{2j+1}^2)$-RIS (see Remark 4.2), (24) tells that $(x_i)_i$ makes $2j+1$ negligible for $(\lambda_i)_i$. From the conclusion of the basic inequality and the estimates on the basis of $T(4n_j, 1/m_j)$, it follows that for every $f \in K_{\omega_1}$

$$|f(\frac{1}{n_{2j+1}} \sum_{i=1}^{n_{2j+1}} \lambda_i x_i)| \le 12(\frac{2}{n_{2j+1}} + \frac{2}{m_{2j+1}^3}) \le \frac{1}{m_{2j+1}^2}, \tag{25}$$

as required. $\square$

Proposition 4.13. *The basis $(e_\alpha)_{\alpha<\omega_1}$ is shrinking and boundedly complete. Therefore $\mathfrak{X}_{\omega_1}$ is reflexive.*

Proof. Since the basis $(e_\alpha)_{\alpha<\omega_1}$ is boundedly complete (see Remark 2.4), we only need to prove that it is also shrinking. Suppose not. Then there exists a strictly increasing sequence $A = \{\alpha_n\}_n$ of ordinals, scalars $(\lambda_n)_n$ and $x^* = w^* - \lim_n \sum_{n=1}^{\infty} \lambda_n e_{\alpha_n}^*$ with $x^* \notin \overline{\langle e_{\alpha_n}^* \rangle_n}$. Thus there exist $\varepsilon > 0$ and successive intervals $(E_n)_n$ such that for all n, $\|E_n x^*\| > \varepsilon$. Choose $(x_n)_n$ in $\mathfrak{X}_A$ with $\operatorname{supp} x_n \subseteq E_n$, $\|x_n\| = 1$ and $x^*(x_n) > \varepsilon$ for all n. It follows that every convex combination $\sum_n \mu_n x_n$ satisfies $\|\sum_n \mu_n x_n\| > \varepsilon$. Now for sufficient large $j \in \mathbb{N}$ we may construct a $(2\varepsilon, 1/n_{2j+1})$-RIS $(y_n)_n$ of ε-normalized averages such that every y_n is an average of $(x_k)_k$. Proposition 4.7 yields that $\|1/(n_{2j}) \sum_{i=1}^{n_{2j}} y_i\| \le (4\varepsilon)/m_{2j} < \varepsilon$, a contradiction. $\square$

5. The operator spaces

In this section we state and prove the main results about operators on $\mathfrak{X}_{\omega_1}$ and its subspaces. The new basic tool is the finite interval representability of a James-like space into $\mathfrak{X}_{\omega_1}$. The section is divided into six subsections. The first concern James like spaces. In the second the finite interval block representability of J_{T_0} is defined and the structure of the space of the step diagonal operators is studied. In the third subsection the spaces $\mathcal{L}(\mathfrak{X}_\gamma)$ are studied and some consequences concerning the structure of the subspaces of $\mathfrak{X}_{\omega_1}$ are obtained. In the fourth subsection the concept of asymptotically equivalent subspaces of $\mathfrak{X}_{\omega_1}$ is introduced and the structure of the spaces $\mathcal{L}(X, \mathfrak{X}_{\omega_1})$ with X subspace of $\mathfrak{X}_{\omega_1}$ is studied. In the fifth subsection a construction of subspaces X is presented such that $\dim \mathcal{L}(X)/\mathcal{S}(X) = 1$ while $\mathcal{L}(X, \mathfrak{X}_{\omega_1})/\mathcal{S}(X, \mathfrak{X}_{\omega_1})$ is of infinite dimension. The last subsection concerns some further results about the operators related to the Fredholm theory of strictly singular operators.

5.1. James-like spaces.

Definition 5.1. Let X be a reflexive space with a 1-subsymmetric basis $(x_n)_n$, and let A be a set of ordinals. $J_X(A)$ is the completion of $(c_{00}(A), \|\cdot\|_{J_X(A)})$, where for $x \in c_{00}(A)$,

$$\|x\|_{J_X(A)} = \sup\{\|\sum_{n=1}^{l}\left(\sum_{i\in I_n} x(i)\right) x_n\|_X \,:\, I_1 < \cdots < I_n \text{ intervals of } A\}.$$

The natural Hamel basis $(v_\alpha)_{\alpha\in A}$ of $c_{00}(A)$ is a bimonotone 1-subsymmetric transfinite basis of $J_X(A)$. Also, for every interval I of A the functional $I^* : J_X(A) \to \mathbb{R}$, $I^*(x) = \sum_{\alpha\in A} x(\alpha)$ belongs to $J_X^*(A)$ and $\|I^*\| = 1$.

Remark 5.2. As we shall see next, ℓ_1 does not embed into $J_X(A)$ and hence the basis $(v_\alpha)_{\alpha\in A}$ is not unconditional.

The following two facts are easy extensions of the corresponding results from [8].

Proposition 5.3. *Let $(y_n)_n$ be a semi-normalized block sequence in $J_X(A)$ with $\sum_{\alpha\in A} y_n(\alpha) = 0$ for every n. Then $(y_n)_n$ is equivalent to the basis $(x_n)_n$ of X.*

Proof. Let $0 < c < C$ be such that $c \le \|y_n\| \le C$ for all n. It is easy to see that:

$$c\|\sum_n a_n x_n\|_X \le \|\sum_n a_n y_n\|_{J_X(A)} \le \sup_{i_1\le i_2\le\cdots\le i_l} \|\sum_{q=1}^{l-1}(|a_{i_q}| + |a_{i_{q+1}}|)x_q\|_X \le$$
$$\le (2CK)\|\sum_n a_n x_n\|_X,$$

where K is the unconditional constant of $(x_n)_n$. The first inequality holds for any block sequence and the second uses our assumptions. □

Corollary 5.4. *The space ℓ_1 does not embed into $J_X(A)$.*

Proof. If not, then from Proposition 1.3 we could find a semi-normalized block sequence $(y_n)_n$ equivalent to the ℓ_1-basis. Therefore, passing if necessary to a further block sequence, we may assume that for all $n \in \mathbb{N}$, $\sum_{\alpha\in A} y_n(\alpha) = 0$. Hence Proposition 5.3 yields that $(y_n)_n$ is equivalent to $(x_n)_n$, a contradiction. $\square$

Remark 5.5. Suppose that A and B are two sets of ordinals with the same order type. Then the unique order-preserving mapping $f : A \to B$ defines naturally an isometry between $\widetilde{f} : J_X(A) \to J_X(B)$ by $\widetilde{f}(\sum_{\alpha\in H} r_\alpha v_\alpha) = \sum_{\alpha\in H} r_\alpha v_{f(\alpha)}$.

The next proposition also extends the corresponding result from [8].

Proposition 5.6. *For every ordinal γ the space $J_X^*(\gamma)$ is generated in norm by $\{[0,\alpha)^*\}_{\alpha<\gamma+1}$.*

Proof. We proceed by induction. It is clear that the successor ordinal case follows immediately from the inductive assumption. So we assume that γ is limit ordinal and for all $\lambda < \gamma$ the conclusion holds. Assume to the contrary that $Y = \overline{\langle [0,\alpha)^* \rangle_{\alpha<\gamma+1}}^{\|\cdot\|} \subsetneq J_X^*(\gamma)$. Then there exists $x^* \in J_X^*(\gamma)$ with $\|x^*\| = 1$ and $\varepsilon > 0$ such that $d(x^*, Y) > \varepsilon$. Observe also that the inductive assumption yields that for all $\alpha < \gamma$ if x^*_α denotes the functional defined by

$$x^*_\alpha(v_\beta) = \begin{cases} 0 & \text{if } \beta < \alpha \\ x^*(v_\beta) & \text{if } \beta \geq \alpha, \end{cases}$$

then $\|x^*_\alpha\| \leq 1$ and $d(x^*_\alpha, Y) > \varepsilon$. In particular for all $\alpha < \gamma$, $d(x^*_\alpha, \langle [\alpha,\gamma)^* \rangle) > \varepsilon$ and from the Hahn-Banach and Goldstine Theorems there exists a finitely supported $\tilde{y}_\alpha \in J_X(\gamma)$ with $\|\tilde{y}_\alpha\| \leq 1$, $\alpha \leq \min\operatorname{supp}\tilde{y}_\alpha$, $x^*(\tilde{y}_\alpha) > \varepsilon$ and $|\sum_{\beta<\gamma} \tilde{y}_\alpha(\beta)| \leq \varepsilon/4$. Assuming further that α is a successor ordinal we consider the vector $y_\alpha = \tilde{y}_\alpha - (\sum_{\beta\geq\alpha} \tilde{y}_\alpha(\beta)) v_{\alpha^-}$. Observe that $\alpha^- \leq \min\operatorname{supp} y_\alpha$, $x^*(y_\alpha) > \varepsilon - \varepsilon/4 > \varepsilon/2$ and $\sum_{\beta<\gamma} y_\alpha(\beta) = 0$. Hence we may inductively choose a block sequence $(z_n)_n$ such that $\varepsilon/2 \leq \|z_n\| \leq 1$, $\sum_{\alpha<\gamma} z_n(\alpha) = 0$ and $x^*(z_n) > \varepsilon/2$. Observe that $(z_n)_n$ is unconditional (Proposition 5.3) therefore equivalent to the ℓ_1-basis which yields a contradiction. $\square$

Corollary 5.7. *For every set of ordinals A we have that* $\dim J_X^*(A) = \#A$. $\square$

5.2. Finite interval representability of J_{T_0} and the space of diagonal operators.

Definition 5.8. Let X and Y be Banach spaces and let $(x_\alpha)_{\alpha<\gamma}$ and $(y_n)_n$ be a transfinite basis for X and a Schauder basis of Y respectively. We say that Y is *finitely interval representable* in X if there exists a constant $C > 0$ such that for every integer n and intervals $I_1 \leq I_2 \leq \cdots \leq I_n$ successive, not necessarily distinct, intervals of γ there exists $z_i \in \langle (x_\alpha)_{\alpha\in I_i} \rangle$ $(i = 1, \ldots, n)$ with $\operatorname{supp} z_1 < \operatorname{supp} z_2 < \cdots < \operatorname{supp} z_n$ and such that the natural order preserving isomorphism $H : \langle (y_i)_{i=1}^n \rangle \to \langle (z_i)_{i=1}^n \rangle$ satisfies $\|H\| \cdot \|H^{-1}\| \leq C$.

Recall that Maurey-Rosenthal [20], in their attempt to solve the unconditional basic sequence problem, have constructed a Banach space X with a weakly-null normalized Schauder basis $(e_n)_n$ having the property that every subsequence of $(e_n)_n$ finitely block represents the James-like space J_{c_0}, or equivalently (and as they said it), every subsequence of $(e_n)_n$ has a arbitrary large finite block subsequence of length k equivalent to the first k-many members of the summing basis of c_0. In our attempt to control non-strictly singular operators on $\mathfrak{X}_{\omega_1}$, we have discovered the following analogous result that surprised us by its powers to explain many phenomena encountered, not only in $\mathfrak{X}_{\omega_1}$, but in essentially any other conditional space constructed so far using the general scheme described above in Section 2. Through all this section γ will denote a limit ordinal.

Theorem 5.9. *Let $(y_\alpha)_{\alpha<\gamma}$ be a normalized transfinite block sequence in $\mathfrak{X}_{\omega_1}$, and Y its closed linear span. Then J_{T_0} is finitely interval representable in the space Y, where T_0 is the mixed Tsirelson space $T[(m_{2j}^{-1}, n_{2j})_j]$.*

We will postpone the proof until Section 8. Throughout all this section $\boldsymbol{C}$ will denote the finitely block representability constant of J_{T_0} in $\mathfrak{X}_{\omega_1}$. We will show in Section 8 that $\boldsymbol{C} < 121$.

Remark 5.10. 1. Let us observe that since, as we will show, the basis of J_{T_0} is not unconditional and it is finitely block representable in any block subsequence of the basis $(e_\alpha)_{\alpha<\omega_1}$, then $\mathfrak{X}_{\omega_1}$ cannot have any unconditional basic sequence. In other words the finite interval representability of J_{T_0} in the block subsequences of $\mathfrak{X}_{\omega_1}$ must make use of the conditional structure of $\mathfrak{X}_{\omega_1}$. Indeed we get more. Suppose that $\mathfrak{X}$ has a transfinite basis, and suppose that a Banach space Y with a conditional basis $(y_n)_n$ is finite block representable in every block sequence of $\mathfrak{X}$ Then $\mathfrak{X}$ does not contain unconditional basic sequences and from Gowers dichotomy [10], $\mathfrak{X}$ is hereditarily indecomposable saturated.
2. The James like space J_{T_0} has the following alternative description. It is the mixed Tsirelson space $T_G[(m_{2j}^{-1}, n_{2j})_j]$, where $G = \{I^* : I \subseteq \mathbb{N} \text{ interval}\}$. The minimal set K_0 of $c_{00}(\mathbb{N})$ which is symmetric, contains G, and is closed under (m_{2j}^{-1}, n_{2j})-operations norms J_{T_0}.

Proposition 5.11. *Let $x_1 < \cdots < x_n$ be finitely supported, $\phi \in K_{\omega_1}$ and set $r_i = \phi x_i$ for each $i = 1, \dots, n$. Then $\| \sum_{i=1}^n r_i v_i \|_{J_{T_0}} \leq \|x_1 + \cdots + x_n\|$.*

Proof. Fix a functional f of K_0 with support contained in $\{1, \dots, n\}$, and a tree-analysis $(f_t)_{t\in\mathcal{T}}$ of f. We show by induction over the tree $\mathcal{T}$ that for every $t \in \mathcal{T}$ there is some $\phi_t \in K_{\omega_1}$ such that $f_t(\sum_{i=1}^n r_i v_i) = \phi_t(x_1 + \cdots + x_n)$. In particular $f_0(\sum_i r_i v_i) = \phi_0(x_1 + \cdots + x_n)$, and hence the desired result holds. If $t \in \mathcal{T}$ is a terminal node, then $f_t = \pm I^*$, $I \subseteq \{1, \dots, n\}$ an interval. We set $\phi_t = \pm\phi|[\min\operatorname{supp} x_{\min I}, \max\operatorname{supp} x_{\max I}]$. It is clear that $\phi_t \in K_{\omega_1}$, and

$$\phi_t(x_1 + \cdots + x_n) = \pm \sum_{i\in I} \phi x_i = \pm \sum_{i\in I} r_i = f_t(\sum_i r_i v_i). \tag{26}$$

If $t \in \mathcal{T}$ is not a terminal node, then $f_t = (1/m_{2j}) \sum_{i=1}^d f_{s_i}$, where $S_t = \{s_1, \dots, s_d\}$ ordered by $f_{s_1} < \cdots < f_{s_d}$. Then $\phi_t = (1/m_{2j}) \sum_{i=1}^d \phi_{s_i}$ clearly satisfies our inductive requirements. □

The next result shows that J_{T_0} is minimal in a precise sense.

Corollary 5.12. *Suppose that X is a Banach space with a normalized Schauder basis $(x_n)_n$ which dominates the summing basis of c_0 and is finitely block represented in $\mathfrak{X}_{\omega_1}$. Then $(x_n)_n$ also dominates the basis $(v_n)_n$ of J_{T_0}.*

Proof. Fix scalars $(a_i)_{i=1}^n$. Choose a normalized block sequence $(w_i)_i^n$ of $\mathfrak{X}_{\omega_1}$ C-equivalent to $(x_i)_{i=1}^n$. Fix $f \in K_0$ with $\operatorname{supp} f \subseteq \{1, \dots, n\}$ and a tree-analysis $(f_t)_{t\in\mathcal{T}}$ of it. We are going to find $\phi_t \in K_{\omega_1}$ such that $|f_t(\sum_{i=1}^n a_i v_i)| \le C|\phi_t(\sum a_i w_i)|$, for each $t \in \mathcal{T}$. This will show that

$$\|\sum_{i=1}^n a_i v_i\|_{J_{T_0}} \le C\|\sum_{i=1}^n a_i w_i\|_{\mathfrak{X}_{\omega_1}} \le C^2\|\sum_{i=1}^n a_i w_i\|_X, \tag{27}$$

as desired. If $t \in \mathcal{T}$ is a terminal node, then $f_t = \pm I^*$, $I \subseteq [1,n]$ interval. Since $(x_n)_n$ dominates the summing basis of c_0, we can find $\phi_t \in K_{\omega_1}$ such that

$$\phi_t(\sum_{i=1}^n a_i w_i) = \|\sum_{i=1}^n a_i w_i\|_{\mathfrak{X}_{\omega_1}} \ge \frac{1}{C}\|\sum_{i=1}^n a_i x_i\|_X \ge \frac{1}{C}|\sum_{i\in I} a_i| = |f_t(\sum_{i=1}^n a_i v_i)|. \tag{28}$$

If t is not terminal node, then we use the appropriate $(m_{2j}^{-1}, n_{2j},)$-operation. $\square$

Definition 5.13. Let $(x_\alpha)_{\alpha<\gamma}$ be a normalized transfinite block sequence, X its closed linear span. We denote by $\mathcal{D}(X)$ the space of all bounded diagonal operators $D : X \to X$ satisfying the property that for all $\alpha < \gamma$ limit there exists some $\lambda_\alpha \in \mathbb{R}$ such that $D(x_\beta) = \lambda_\alpha x_\beta$ for every $\beta \in [\alpha, \alpha+\omega)$. We also denote by $\widetilde{\mathcal{D}}(X)$ the space of all diagonal operators (not necessarily bounded) satisfying the above condition acting on $\langle x_\alpha\rangle_{\alpha<\gamma}$.

Notice the following (linear) decomposition of $\langle x_\alpha\rangle_{\alpha<\gamma}$,

$$\langle x_\alpha\rangle_{\alpha<\gamma} = \bigoplus_{\alpha\in\Lambda(\gamma)} \langle x_\beta\rangle_{\beta\in[\alpha,\alpha+\omega)}. \tag{29}$$

The *canonical decomposition* of $y \in \langle x_\alpha\rangle_{\alpha<\gamma}$ in X is $y = y_1 + \cdots + y_n$ given by (29).

Remark 5.14. $\mathcal{D}(X)$ is a closed subalgebra of $\mathcal{L}(X)$.

For an ordinal μ we denote by $\Lambda(\mu)$ the set of limit ordinals $< \mu$, and by $\Lambda(\mu)^{(0)}$ the set of limit ordinals $\alpha = \beta + \omega < \mu$ with $\beta \in \Lambda(\mu)$. We denote this (unique) β by α^-. Notice that $\Lambda(\mu)^{(0)}$ is the set of isolated points of $\Lambda(\mu)$ with respect to the order-topology. For technical reasons, 0 is considered as limit ordinal.

Remark 5.15. Notice that for γ a limit ordinal, $\Lambda(\gamma+1)^{(0)}$ is order isomorphic to $\Lambda(\gamma)$ via the predecessor map.

Definition 5.16. Let $D \in \widetilde{\mathcal{D}}(X)$. We define the map $\xi_D : \Lambda(\gamma+1)^{(0)} \to \mathbb{R}$ by

$$D(x_{\alpha^-}) = \xi_D(\alpha)x_{\alpha^-}. \tag{30}$$

Namely, $\xi_D(\alpha)$ is the eigenvalue of D associated to the eigenvectors $(x_\beta)_{\beta\in[\alpha^-,\alpha)}$.

We consider the following linear map $\Xi : \widetilde{D}(X) \to c_{00}(\Lambda(\gamma+1)^{(0)})^\#$ defined by

$$\Xi(D)(v_\alpha) = \xi_D(\alpha), \tag{31}$$

where $c_{00}(\Lambda(\gamma+1)^{(0)})^\#$ denotes the algebraic conjugate of $c_{00}(\Lambda(\gamma+1)^{(0)}$. The main goal here is to show that Ξ defines an isomorphism between $\mathcal{D}(X)$ and $J^*_{T_0}(\Lambda(\gamma+1)^{(0)})$. For $D \in \widetilde{\mathcal{D}}(X)$, let us denote

$$\|D\| = \sup\{\|Dx\|_{\mathfrak{X}_{\omega_1}} : x \in \langle x_\alpha\rangle_{\alpha<\gamma}, \|x\|_{\mathfrak{X}_{\omega_1}} \le 1\} \le \infty,$$

and for $f \in c_{00}(\Lambda(\gamma+1)^{(0)})^\#$,

$$\|f\| = \sup\{f(x) : x \in c_{00}(\lambda(\gamma+1)^{(0)}), \|x\|_{J_{T_0}} \le 1\} \le \infty.$$

Proposition 5.17. *$\|D\| \leq \|\Xi(D)\| \leq \boldsymbol{C}\|D\|$ for every $D \in \widetilde{\mathcal{D}}(X)$.*

Proof. Fix $D \in \widetilde{\mathcal{D}}(X)$, and $\varepsilon > 0$. Let $y \in \langle x_\alpha \rangle_{\alpha<\gamma}$ with $\|y\| \leq 1$ be such that $|\|D\| - \|Dy\|| < \varepsilon$. Let $y = y_1 + \cdots + y_n$ be the canonical decomposition of y in X, and $\alpha_1, \dots, \alpha_n$ be such that $y_i \in \langle x_\beta \rangle_{\beta \in [\alpha_i^-, \alpha_i)}$ for every $1 \leq i \leq n$. Let $\phi \in K$ be such that $\|Dy\| = \phi(Dy)$, and set $r_i = \phi y_i$ for $i = 1, \dots, n$. By Proposition 5.11, $\|\sum_{i=1}^n r_i v_i\|_{J_{T_0}} \leq \|x\|$, and since $(v_\alpha)_\alpha$ is 1-subsymmetric we have that $\|\sum_{i=1}^n r_i v_{\alpha_i}\|_{J_{T_0}} \leq \|y\| \leq 1$. Hence

$$\|\Xi(D)\| \geq \|\Xi(D)(\sum_{i=1}^n r_i v_{\alpha_i})\|_{J_{T_0}} = \|\sum_{i=1}^n \xi_D(\alpha_i) r_i v_{\alpha_i}\|_{J_{T_0}} \geq \sum_{i=1}^n \xi_D(\alpha_i) = \phi(Dy) \geq \|D\| - \varepsilon. \tag{32}$$

This shows that $\|D\| \leq \|\Xi(D)\|$. Fix $v = \sum_{i=1}^n a_i v_{\alpha_i} \in J_{T_0}$ with $\|v\|_{J_{T_0}} \leq 1$, and choose a finite normalized block sequence $(w_i)_{i=1}^n$ $\boldsymbol{C}$-equivalent to $(v_{\alpha_i})_{i=1}^n$ with $w_i \in \langle x_\beta \rangle_{\beta \in [\alpha_i^-, \alpha_i)}$ for every $i = 1, \dots, n$ (indeed we may assume that the natural isomorphism $F : \langle w_i \rangle_{i=1}^n \to \langle v_i \rangle_{i=1}^n$ satisfies that $\|F\| \leq 1$, $\|F^{-1}\| \leq \boldsymbol{C}$; see Corollary 8.15). Then,

$$\begin{aligned}\|\Xi(D)(v)\|_{J_{T_0}} &= \|\sum_{i=1}^n \xi_D(\alpha_i) a_i v_{\alpha_i}\|_{J_{T_0}} \leq \|\sum_{i=1}^n \xi_D(\alpha_i) a_i w_i\|_{\mathfrak{X}_{\omega_1}} = \|D(\sum_{i=1}^n a_i w_i)\|_{\mathfrak{X}_{\omega_1}} \leq \\ &\leq \|D\| \|\sum_{i=1}^n a_i w_i\|_{\mathfrak{X}_{\omega_1}} \leq \boldsymbol{C}\|D\|.\end{aligned} \tag{33}$$

□

Theorem 5.18. *The spaces $\mathcal{D}(X)$ and $J_{T_0}^*(\Lambda(\gamma+1)^{(0)})$ are isomorphic.*

Proof. By Proposition 5.17, $\Xi|\mathcal{D}(X) : \mathcal{D}(X) \to J_{T_0}^*(\Lambda(\gamma+1)^{(0)})$ is an isomorphism. To see that it is also onto consider $f \in J_{T_0}^*(\Lambda(\gamma+1)^{(0)})$ and define $D_f \in \widetilde{\mathcal{D}}(X)$ as follows. For $\beta \in [\alpha^-, \alpha)$ set $D_f(x_\beta) = f(v_\alpha) x_\beta$. It is easy to check that $\Xi(D_f) = f$. This completes the proof. □

Corollary 5.19. *Let X and Y be the closed linear span of two transfinite block sequences of the same length γ. Then the natural mapping $\psi_\gamma : \mathcal{D}(X) \to \mathcal{D}(Y)$ defined by $\psi_\gamma(D) = D_{\xi_D}$ is an isomorphism.* □

Our intention now is to compare $\mathcal{D}(X)$ and $\mathcal{D}(\mathfrak{X}_{\omega_1})$.

Definition 5.20. 1. Given a closed $A \subseteq \Lambda(\omega_1 + 1)$, let $\widetilde{\mathcal{D}}_A(\mathfrak{X}_{\omega_1})$ be the subalgebra of $\widetilde{\mathcal{D}}(\mathfrak{X}_{\omega_1})$ consisting on all $D \in \mathcal{D}(\mathfrak{X}_{\omega_1})$ satisfying that for every $\alpha \in A^{(0)}$, there is some λ_α such that $D|\mathfrak{X}_{[\alpha^-,\alpha)} = \lambda_\alpha i_{\mathfrak{X}_{[\alpha^-,\alpha)}, \mathfrak{X}_{\omega_1}}$ and $D|\mathfrak{X}_{[\max A, \omega_1)} = 0$. Let $\mathcal{D}_A(\mathfrak{X}_{\omega_1})$ be the subalgebra of bounded operators of $\widetilde{\mathcal{D}}_A(\mathfrak{X}_{\omega_1})$.
2. Given a transfinite block sequence $(x_\alpha)_{\alpha<\gamma}$, let $\Gamma_X \subseteq \Lambda(\omega_1 + 1)$ be defined as follows. Let

$$\Gamma' = \{\sup_{n\to\infty} \max \operatorname{supp} x_{\alpha_n} \, : \, (\alpha_n)_n \uparrow, \alpha_n < \gamma\}, \tag{34}$$

and let $\Gamma_X = \Gamma' \cup \{0, \sup \Gamma'\}$. Another interpretation of Γ_X is to consider the map $f_X : \Lambda(\gamma+1) \to \omega_1$ defined by $f_X(\alpha) = \sup_{\beta<\alpha} \max \operatorname{supp} x_\beta$ and Γ_X is nothing else but the image $f(\Lambda(\gamma+1))$, and hence $\Gamma_X \setminus \max\{\Gamma_X\}$ and $\Lambda(\gamma+1)^{(0)}$ are order isomorphic.

3. Given $D \in \mathcal{D}(X)$, let $E(D) \in \widetilde{\mathcal{D}}_{\Gamma_X}(\mathfrak{X}_{\omega_1})$ be the unique extension of D. Notice that $D|X \in \mathcal{D}(X)$ for every $D \in \mathcal{D}_{\Gamma_X}(\mathfrak{X}_{\omega_1})$.

Theorem 5.21 (*Extension Theorem*). *For every $X \hookrightarrow \mathfrak{X}_{\omega_1}$ generated by a transfinite block sequence the following hold:*
(a) Every $D \in \mathcal{D}(X)$ is extended to a step diagonal operator ED in $\mathcal{D}(\mathfrak{X}_{\omega_1})$.
(b) The map $D \mapsto ED$ defines a linear isomorphism from $\mathcal{D}(X)$ onto the space $\mathcal{D}_{\Gamma_X}(\mathfrak{X}_{\omega_1})$.

Proof. We show that $\|E(D)\| \leq \boldsymbol{C}\|D\|$ for every $D \in \mathcal{D}(X)$. Fix a finitely supported $y \in \mathfrak{X}_{\omega_1}$ such that $\|y\| \leq 1$ and $\|E(D)\| = \|E(D)(y)\|$. Since $\mathcal{I} = \{[\alpha^-_{\Gamma_X}, \alpha) : \alpha \in \Gamma_X^{(0)}\} \cup \{[\max \Gamma_X, \omega_1)\}$ is a partition of ω_1, y has a unique decomposition $y = y_1 + \cdots + y_n$ for $I_1 < \cdots < I_n$ in $\mathcal{I}$ and $y_i \in \langle e_\alpha \rangle_{\alpha \in I_i}$. Notice that $E(D)|\mathfrak{X}_{[\max \Gamma_X, \omega_1)} = 0$, so we may assume that $I_n \neq [\max \Gamma_X, \omega_1)$. By definition of $E(D)$ we have that $E(D)(y) = \sum_{i=1}^n \xi_D(\beta_i) y_i$ where $\beta_i = f_X^{-1}(\alpha_i)$ for every $i = 1, \dots, n$. Choose $\phi \in K_{\omega_1}$ such that $\|E(D)(y)\| = \phi(E(D)(y))$. By Proposition 5.17,

$$\begin{aligned}\|E(D)\| =&\phi(\sum_{i=1}^n \xi_D(\beta_i) y_i) = \sum_{i=1}^n \xi_D(\beta_i)\phi(y_i) = \Xi(D)(\sum_{i=1}^n \phi(y_i) v_{\beta_i}) \leq \\ \leq&\|\Xi(D)\|_{J^*_{T_0}(\Lambda(\gamma+1)^{(0)})} \|\sum_{i=1}^n \phi(y_i) v_i\|_{J_{T_0}} \leq \boldsymbol{C}\|D\|. \end{aligned} \tag{35}$$

□

5.3. The spaces $\mathcal{L}(\mathfrak{X}_\gamma)$.

Definition 5.22. A sequence $(x_1, \phi_1, \dots, x_{n_{2j+1}}, \phi_{n_{2j+1}})$ is called a $(0, j)$-*dependent sequence* if the following conditions are fulfilled:

DS0.1 $\Phi = (\phi_1, \dots, \phi_{n_{2j+1}})$ is a $2j+1$-special sequence and $\phi_i x_{i'} = 0$ for every $1 \leq i, i' \leq n_{2j+1}$.
DS0.2 There exists $\{\psi_1, \dots, \psi_{n_{2j+1}}\}$ such that $w(\psi_i) = w(\phi_i)$, $\#\mathrm{supp}\, x_i \leq w(\phi_{i+1})/n^2_{2j+1}$ and (x_i, ψ_i) is a $(6, 2j_i)$-exact pair for every $1 \leq i \leq n_{2j+1}$.
DS0.3 If $H = (h_1, \dots, h_{n_{2j+1}})$ is an arbitrary $2j+1$-special sequence, then

$$\left(\bigcup_{\kappa_{\Phi,H} < i < \lambda_{\Phi,H}} \mathrm{supp}\, x_i \right) \cap \left(\bigcup_{\kappa_{\Phi,H} < i < \lambda_{\Phi,H}} \mathrm{supp}\, h_i \right) = \emptyset. \tag{36}$$

Proposition 5.23. *For every $(0, j)$-dependent sequence $(x_1, \phi_1, \dots, x_{n_{2j+1}}, \phi_{n_{2j+1}})$ we have that*

$$\|\frac{1}{n_{2j+1}}(x_1 + \cdots + x_{n_{2j+1}})\| \leq \frac{1}{m^2_{2j+1}}.$$

Proof. The proof is rather similar to the proof of Proposition 3.6. One first shows that $|\psi(1/n_{2j+1} \sum_{i \in E} x_i)| \leq 12(1 + \#E/n^2_{2j+1})$ for every special functional ψ with $w(\psi) = m_{2j+1}$, and then the result follows from the basic inequality, since, by condition (*DS0.2*), $(x_i)_{i=1}^{n_{2j+1}}$ is a $(12, 1/n^2_{2j+1})$-RIS. □

Proposition 5.24. *Suppose that $(y_k)_k$ is a (C, ε)-RIS, and suppose that $T : \langle y_k \rangle_k \to \mathfrak{X}_{\omega_1}$ is a linear function (not necessarily bounded) such that $\lim_{n \to \infty} d(Ty_n, \mathbb{R}y_n) \neq 0$. Then for every $\varepsilon > 0$ there is some $z \in \langle y_k \rangle_k$ such that $\|z\| < \varepsilon \|Tz\|$.*

Proof. We may assume that there is some $\delta > 0$ such that $\inf_n d(Ty_n, \mathbb{R}y_n) > \delta > 0$, and also that $(Ty_n)_n$ is a block sequence (hint: Consider the following limit ordinal

$$\gamma_0 = \min\{\gamma < \omega_1 \,:\, \exists A \in [\mathbb{N}]^\infty \inf_{n\in A} d(P_\gamma T y_n, \mathbb{R}y_n) > 0\}, \tag{37}$$

pass to a subsequence of $(y_n)_n$ and replace T by $P_{\gamma_0}T$).

Claim. *There exist an infinite set $A \subseteq \mathbb{N}$ and a block sequence $(f_n)_{n\in A}$ of functionals in K_{ω_1} such that:*
(a) *For every $n \in A$, $f_nTy_n \geq \delta$, $f_ny_n = 0$, $\operatorname{ran} f_n \subseteq \operatorname{ran} Ty_n$ and $\operatorname{supp} f_n \cap \operatorname{supp} y_m = \emptyset$ for every $m \neq n$.*
(b) *Either for every $n \in A$ $\max\operatorname{supp} y_n \geq \max\operatorname{supp} f_n$ or for every $n \in A$ $\max\operatorname{supp} y_n \leq \max\operatorname{supp} f_n$.*

Proof of Claim: By the Hahn-Banach theorem, for each $n \in \mathbb{N}$ we can find a functional f_n of norm 1 such that $f_n(Ty_n) \geq \delta$ and $f_n(y_n) = 0$. Since the w^*-closure of K_{ω_1} is $B_{\mathfrak{X}^*_{\omega_1}}$ (notice that K by definition is closed under rational convex combinations) and K_{ω_1} is closed under restriction over intervals, we may assume that $f_n \in K_{\omega_1}$ and $\operatorname{ran} f_n \subseteq \operatorname{ran} Ty_n$. Let $\alpha = \max_n \operatorname{supp} y_n$ and $\beta = \max_n \operatorname{supp} f_n$. If $\alpha \neq \beta$, it is rather easy to achieve the desired result. If $\alpha = \beta$, then we can pass to a subsequence A and distort f_n such that for every $n \in A$, $\max\operatorname{supp} f_n \geq \max\operatorname{supp} y_n$. □

So, we may assume that $(f_n)_n$ satisfies the requirements of previous Claim. Fix j with $m_{2j+1} > 12/(\varepsilon\delta)$.

Claim. *There is a $(0, j)$-dependent sequence $(z_1, \phi_1, \dots, z_{n_{2j+1}}, \phi_{n_{2j+1}})$ such that for every $k \leq n_{2j+1}$, $z_k \in X$, $\operatorname{ran}\phi_k \subseteq \operatorname{ran} Tz_k$ and $\phi_k Tz_k > \delta$.*

Proof of Claim: Choose j_1 even such that $m_{2j_1} > n^2_{2j+1}$, and choose $F_1 \subseteq \mathbb{N}$ of size n_{2j_1} such that $(y_k)_{k\in F_1}$ is a $(3, 1/n^2_{2j_1})$-RIS (going to a subsequence of $(y_k)_k$; see Remark 4.2). Set

$$\phi_1 = \frac{1}{m_{2j_1}} \sum_{i\in F_1} f_i \in K_{\omega_1} \text{ and } z_1 = \frac{m_{2j_1}}{n_{2j_1}} \sum_{k\in F_1} y_k.$$

Notice that $\phi_1Tz_1 = (1/n_{2j_1})\sum_{k\in F_1} f_kTy_k > \delta$ and by (a) from the Claim, we have that $\phi_1z_1 = (1/n_{2j_1})\sum_{k\in F_1}\sum_{l\in F_1} f_k(y_l) = 0$. Pick

$$p_1 \geq \max\{p_\varrho(\operatorname{supp} z_1 \cup \operatorname{supp} Tz_1 \cup \operatorname{supp}\phi_1), \#\operatorname{supp} z_1 \cdot n^2_{2j+1}\} \tag{38}$$

and set $2j_2 = \sigma_\varrho(\Phi_1, m_{2j_1}, p_1)$. Now choose $F_2 > F_1$ finite of length n_{2j_2} such that $(x_k)_{k\in F_2}$ is a $(3, 1/n^2_{2j_2})$-RIS. Set

$$\phi_2 = \frac{1}{m_{2j_2}} \sum_{k\in F_2} f_k \in K_{\omega_1} \text{ and } z_2 = \frac{m_{2j_2}}{n_{2j_2}} \sum_{k\in F_2} y_k. \tag{39}$$

Notice that $\phi_2 > \phi_1$, $\phi_2Tz_2 > \delta$ and $\phi_2z_2 = 0$. Pick

$$p_2 \geq \max\{p_1, p_\varrho(\operatorname{supp} z_1 \cup \operatorname{supp} z_2 \cup \operatorname{supp} Tz_1 \cup \operatorname{supp} Tz_2 \cup \operatorname{supp}\Phi_1 \cup \operatorname{supp}\Phi_2), \#\operatorname{supp} z_2 \cdot n^2_{2j+1}\}\} \tag{40}$$

and set $2j_3 = \sigma_\varrho(\phi_1, m_{2j_1}, p_1, \phi_2, m_{2j_2}, p_2)$, and so on. Let us check that $(z_1, \phi_1, \dots, z_{n_{2j+1}}, \phi_{n_{2j+1}})$ is a $(0,j)$-dependent sequence: Condition (*DS0.1*) and (*DS0.2*) are rather easy to check from the definition of this sequence. Let us check (*DS0.3*). There are two cases: (a) Suppose that $\max\operatorname{supp} z_k \le \max\operatorname{supp}\phi_k$ for every $1 \le k \le n_{2j+1}$. Then $\operatorname{supp} z_k \subseteq \operatorname{supp}\overline{\phi_{\lambda_{\Phi,H}-1}}^{p_{\lambda_{\Phi,H}-1}}$ for every $\kappa_{\Phi,H} < k < \lambda_{\Phi,H}$. Then part 2 of (TP.3) gives the desired result. (b) Suppose that $\max\operatorname{supp}\phi_k \le \max\operatorname{supp} z_k$ for every $1 \le k \le n_{2j+1}$. Then $\operatorname{supp}\phi_k \subseteq \operatorname{supp}\overline{z_{\lambda_{\Phi,H}-1}}^{p_{\lambda_{\Phi,H}-1}}$ for every $\kappa_{\Phi,H} < k < \lambda_{\Phi,H}$, and we are done by part 1 of (TP.3). □

Fix a $(0,j)$-dependent sequence $(z_1, \phi_1, \dots, z_n, \phi_{n_{2j+1}})$ as in the Claim, and set

$$z = \frac{1}{n_{2j+1}} \sum_{k=1}^{n_{2j+1}} (-1)^{k+1} z_k \text{ and } \phi = \frac{1}{m_{2j+1}} \sum_{k=1}^{n_{2j+1}} \phi_k.$$

Then $\phi T z = 1/n_{2j+1} \sum_{k=1}^{n_{2j+1}} (-1)^{k+1} \phi T z_k \ge \delta/m_{2j+1}$ and $\|z\| \le 12/m_{2j+1}^2$. So, $\|T(z)\| \ge \delta/m_{2j+1} \ge \delta m_{2j+1}\|z\|/12 > \varepsilon\|z\|$ as desired. □

Corollary 5.25. *Let $(y_k)_k$ be a (C,ε)-RIS, Y its closed linear span and $T : Y \to \mathfrak{X}_{\omega_1}$ be a bounded operator. Then $\lim_{n\to\infty} d(Ty_k, \mathbb{R}y_k) = 0$.*

Proof. If not, by the previous Proposition 5.24, we can find a vector $z \in \langle y_k\rangle_k$ such that $\|z\| < (1/\|T\|)\|Tz\|$ which is impossible if T is bounded. □

Lemma 5.26. *Let $(x_n)_n$ be a (C,ε)-RIS, X its closed span and $T : X \to \mathfrak{X}_{\omega_1}$ be a bounded operator. Then $\lambda_T : \mathbb{N} \to \mathbb{R}$ defined by $d(Tx_n, \mathbb{R}x_n) = \|Tx_n - \lambda_T(n)x_n\|$ is a convergent sequence.*

Proof. Fix any two strictly increasing sequences $(\alpha_n)_n$ and $(\beta_n)_n$ with $\sup_n \alpha_n = \sup_n \beta_n$, and suppose that $\lambda_T(\alpha_n) \to_n \lambda_1$, $\lambda_T(\beta_n) \to_n \lambda_2$. By going to a subsequences, we can assume that $x_{\alpha_n} < x_{\beta_n}$ for every n. Since the closed linear span of $\{x_{\alpha_n}\}_n \cup \{x_{\beta_n}\}_n$ is an H.I. space, we can find for every ε two normalized vectors $w_1 \in \langle x_{\alpha_n}\rangle_n$ and $w_2 \in \langle x_{\beta_n}\rangle_n$ such that $\|Tw_1 - \lambda_1 w_1\| \le \varepsilon/3$, $\|Tw_2 - \lambda_2 w_2\| \le \varepsilon/3$ and $\|w_1 - w_2\| \le \varepsilon/3\|T\|$. Then we have that

$$\|\lambda_1 w_1 - \lambda_2 w_2\| \le \|Tw_1 - \lambda_1 w_1\| + \|Tw_1 - Tw_2\| + \|Tw_2 - \lambda_2 w_2\| \le \varepsilon, \tag{41}$$

and hence,

$$\varepsilon \ge \|\lambda_1 w_1 - \lambda_2 w_2\| \ge |\lambda_1 - \lambda_2|\|w_1\| - |\lambda_2|\|w_1 - w_2\| \ge |\lambda_1 - \lambda_2| - |\lambda_2|\varepsilon. \tag{42}$$

So, $|\lambda_1 - \lambda_2| \le \varepsilon(1 + |\lambda_2|)$ for every ε. This implies that $\lambda_1 = \lambda_2$. □

Definition 5.27. Recall that for a set A of ordinals $A^{(0)}$ is the set of isolated points of A. Fix a transfinite block sequence $(x_\alpha)_{\alpha<\gamma}$, let X be the closed linear span of it and let $T : X \to \mathfrak{X}_{\omega_1}$ be a bounded operator. We define the *step function* ξ_T of T $\xi_T : \Lambda(\gamma+1)^{(0)} \to \mathbb{R}$ as follows: Let γ be a successor limit ordinal less than γ. Let $\xi_T(\gamma) = \xi \in \mathbb{R}$ be such that $\lim_{n\to\infty} \|Ty_n - \xi y_n\| = 0$ for every $(3,\varepsilon)$-RIS $(y_n)_n$ satisfying that $\sup_n \max\operatorname{supp} y_n = \gamma$. Lemma 5.26 shows that ξ exists and is unique, and that ξ_T can be extended to a continuous $\widetilde{\xi_T} : \Lambda(\gamma+1) \to \mathbb{R}$.

Given a mapping $\xi : \Lambda(\gamma+1)^{(0)} \to \mathbb{R}$ we define the diagonal, not necessarily bounded, operator $D_\xi : X \to X$ in the natural way by $D_\xi(x_\alpha) = \xi(\alpha+\omega)x_\alpha$. Given a bounded $T : X \to \mathfrak{X}_{\omega_1}$ we define the *diagonal step operator* $D_T : \langle x_\alpha\rangle_{\alpha<\gamma} \to \mathfrak{X}_{\omega_1}$ of T as $D_T = D_{\xi_T}$.

REMARK 5.28. The function ξ_T has only countable many values. This follows from the fact that it can be extended to a continuous function $\widetilde{\xi}_T$ defined on $\Lambda(\gamma+1)$. As it is well known, if $\gamma=\omega_1$ the function $\widetilde{\xi}_T$ is eventually constant.

Proposition 5.29. *The sequence $(\|(T-D_T)(y_n)\|)_n \in c_0(\mathbb{N})$ for every RIS $(y_n)_n$ in X.*

PROOF. This is just a consequence of the definition of D_T. $\square$

Proposition 5.30. *A bounded operator $T: X \to \mathfrak{X}_{\omega_1}$ is strictly singular iff $\xi_T = 0$.*

PROOF. Suppose that T is not strictly singular. Then there is a block sequence $(y_n)_n$ such that T is an isomorphism restricted to the closed linear span Y of $(y_n)_n$. Going to a block subsequence if necessary, we assume that $(y_n)_n$ is a RIS. Since $T|Y$ is an isomorphism, $\lim_{n\to\infty}\|Ty_n\|>0$. This implies that $\xi_T|\Lambda(\alpha+1)^{(0)} \neq 0$, since otherwise $\widetilde{\xi}_T(\alpha)=0$ contradicting the above inequality.

Suppose now that $\xi_T \neq 0$. Choose some successor limit γ such that $\xi_T(\gamma)\neq 0$. Then we can find a block sequence $(y_n)_n \subseteq X_\gamma$ such that T is close enough to $\xi_T(\gamma) i_{Y,\mathfrak{X}_{\omega_1}}$, where Y is the closed linear span of $(y_n)_n$. Hence, T is not strictly singular. $\square$

Proposition 5.31. *Let $(x_\alpha)_{\alpha<\gamma}$ be a transfinite block sequence, X its closed linear span of $(x_\alpha)_{\alpha<\gamma}$ and a bounded operator $T: X\to \mathfrak{X}_{\omega_1}$. Then $\|D_T\| \leq \boldsymbol{C}\|T\|$ and hence $D_T \in \mathcal{D}(X)$.*

PROOF. Fix a normalized $y \in \langle x_\alpha\rangle_{\alpha<\gamma}$. Let $y = y_1+\cdots+y_n$ be its decomposition in X, $y_i \in \langle x_\beta\rangle_{\beta\in[\alpha_i^-,\alpha_i)}$ for $i=1,\ldots,n$. Choose $\phi \in K_{\omega_1}$ such that $\phi(D(y)) = \|D(y)\|$. Then,

$$\|D(y)\| = \sum_{i=1}^n \xi_T(\alpha_i)\phi(y_i) = (\sum_{i=1}^n \xi_T(\alpha_i)v_i^*)(\sum_{i=1}^n \phi(y_i)v_i) \leq \|\sum_{i=1}^n \xi_T(\alpha_i)v_i\|_{J_{T_0}}, \tag{43}$$

the last inequality holding because $\|\sum_{i=1}^n \phi(y_i)v_{\alpha_i}\|_{J_{T_0}} \leq \|y\|_{\mathfrak{X}_{\omega_1}} \leq 1$. We finish with the next claim.

Claim. $\|\sum_{i=1}^n \xi_T(\alpha_i)v_i^*\|_{J_{T_0}^*} \leq \boldsymbol{C}\|T\|$.

Proof of Claim: Fix $\varepsilon>0$. By the finitely block representability of J_{T_0} in $\mathfrak{X}_{\omega_1}$ and Proposition 5.29 we can produce inductively $w_1,\ldots,w_n$ such that *(1)* $w_i \in \langle x_\beta\rangle_{\beta\in[\alpha_i^-,\alpha_i)}$,
(2) the natural isomorphism $F: \langle w_i\rangle_{i=1}^n \to \langle v_i\rangle_{i=1}^n$ is such that $\|F\|\leq 1$ and $\|F^{-1}\| \leq \boldsymbol{C}$, and
(3) $\sum_{i=1}^n \|\xi_T(\alpha_i)w_i - Tw_i\| < \varepsilon$.

Choose $x=\sum_{i=1}^n r_i v_i \in J_{T_0}$ of norm 1 such that $\|\sum_{i=1}^n \xi_T(\alpha_i)v_i^*\|_{J_{T_0}^*} = \sum_{i=1}^n \xi_T(\alpha_i) r_i$. Then $\|\sum_{i=1}^n r_i w_i\|_{\mathfrak{X}_{\omega_1}} \leq \boldsymbol{C}$ and hence

$$\|D_T(\sum_{i=1}^n r_i w_i)\| \geq \|\sum_{i=1}^n r_i \xi_T(\alpha_i) v_i\|_{J_{T_0}} \geq \sum_{i=1}^n \xi_T(\alpha_i) r_i = \|\sum_{i=1}^n \xi_T(\alpha_i)v_i^*\|_{J_{T_0}^*}. \tag{44}$$

This implies that $\|\sum_{i=1}^n \xi_T(\alpha_i)v_i^*\|_{J_{T_0}^*} \leq \|T(\sum_{i=1}^n r_i w_i)\| + \|(T-D_T)(\sum_{i=1}^n r_i w_i)\| \leq \boldsymbol{C}\|T\| + \varepsilon$. $\square$

$\square$

Theorem 5.32. *Let $(x_\alpha)_{\alpha<\gamma}$ be a normalized block sequence of $\mathfrak{X}_{\omega_1}$, X its closed linear span. Then for every bounded operator $T: X\to\mathfrak{X}_{\omega_1}$, $D_T : X \to \mathfrak{X}_{\omega_1}$ is bounded and $T-D_T$ is strictly singular.*

Proof. This follows from Proposition 5.30 and Proposition 5.31. $\square$

Corollary 5.33. *Any bounded operator from the closed linear span X of a transfinite block sequence into the space $\mathfrak{X}_{\omega_1}$ is the sum of the restriction of a unique diagonal step operator $D \in \mathcal{D}_X(\mathfrak{X}_{\omega_1})$ and an strictly singular operator.*

Proof. This follows from the previous theorem and Theorem 5.21. $\square$

Corollary 5.34. (1) *For $T : X \to \mathfrak{X}_{\omega_1}$ bounded TFAE: (a) T is strictly singular, (b) $\xi_T = 0$, and (c) $D_T = 0$.*
(2) *The transformation $T \mapsto D_T$ is a projection in the operator algebra $\mathcal{L}(X)$ of norm $\leq \boldsymbol{C}$.* $\square$

Proposition 5.35. *Let $X \hookrightarrow \mathfrak{X}_{\omega_1}$, $I \subseteq \omega_1$ an interval such that $P_I|X$ is not strictly singular. Then for every $\varepsilon > 0$ there exist a normalized sequence $(x_n)_n$ in X and a normalized block sequence $(z_n)_n$ in $\mathfrak{X}_I$ such that $\sum_n \|y_z - z_n\| < \varepsilon$.*

Proof. Set $I = [\alpha, \beta]$ and suppose that $P_I|X$ is not strictly singular. Let

$$\gamma_0 = \{\gamma \in (\alpha, \beta] \,:\, P_\gamma|X \text{ is not strictly singular}\}.$$

We can find for every $\varepsilon > 0$, $(y_n)_n \subseteq X$ and a block sequence $(w_n)_n \subseteq \mathfrak{X}_{\gamma_0}$ such that P_{γ_0} is an isomorphism when restricted to the closed linear span of $(y_n)_n$, $\sup_n \max\operatorname{supp} w_n = \gamma_0$ and $\sum_n \|w_n - P_{\gamma_0} y_n\| \leq \varepsilon/2$. Consider $U : \overline{\langle w_n \rangle_n} \to \mathfrak{X}_{[\gamma_0,\omega_1)}$ defined by $U w_n = P_{[\gamma_0,\omega_1)} y_n$. Notice that U is bounded. Since $\xi_U = 0$, U is strictly singular. Hence we can find a block sequence $(z_n)_n$ of $(w_n)_n$ such that for all n, $\|U z_n\| \leq \varepsilon/2^{n+1}$ and hence the corresponding block sequence $(x_n)_n$ of $(y_n)_n$ satisfies that $\sum_n \|z_n - x_n\| \leq \varepsilon$. Finally, notice that for large enough n_0, $(z_n)_{n \geq n_0} \subseteq \mathfrak{X}_I$. $\square$

Corollary 5.36. *The space $\mathfrak{X}_{\omega_1}$ is arbitrarily distortable.*

Proof. For $j \in \mathbb{N}$, and $x \in \mathfrak{X}_{\omega_1}$, let $\|x\|_{2j} = \sup\{\phi(x) \,:\, w(\phi) = m_{2j}\}$. Let $X \hookrightarrow \mathfrak{X}_{\omega_1}$. Since for every $\varepsilon > 0$ we can find a subspace of X generated by a Schauder basis $(y_n)_n$ and a normalized block sequence $(z_n)_n$ of $\mathfrak{X}_{\omega_1}$ such that $\sum_n \|y_n - z_n\| \leq \varepsilon$, without loss of generality we can assume that X is generated by a block sequence $(z_n)_n$. Now, we can find an $(6, j)$-exact pair (x, ϕ), with $x \in \langle z_n \rangle_n$ and hence $1 \leq \|x\|_{2j} \leq \|x\| \leq 6$. And for any other $j' > j$, a $(6, 2j')$-exact pair (x', ϕ') with $x' \in \langle z_n \rangle_n$ and hence $1 \leq \|x'\| \leq 6$ and $\|x'\|_{2j} \leq 12/m_{2j}$. So,

$$\frac{\|x/\|x\|\|_{2j}}{\|x'/\|x'\|\|_{2j}} \geq \frac{1/6}{12/m_{2j+1}} = \frac{m_{2j+1}}{72}. \tag{45}$$

$\square$

Definition 5.37. Two Banach spaces X and Y are called totally incomparable if and only if no infinite dimensional closed $X_1 \hookrightarrow X$ is isomorphic to $Y_1 \hookrightarrow Y$.

Corollary 5.38. *For disjoint infinite intervals I and J, the spaces $\mathfrak{X}_I$ and $\mathfrak{X}_J$ are totally incomparable.*

Proof. Suppose not, and let $X \hookrightarrow \mathfrak{X}_I$, and $Y \hookrightarrow \mathfrak{X}_J$ such that $T : X \to Y$ is an onto isomorphism. By the previous Proposition 5.35, we can assume that X is generated by a block sequence. But since $\xi_T = 0$, T cannot be isomorphism. This is a contradiction. $\square$

Another consequence of the representability of J_{T_0} on each transfinite block sequence is that we can identify the space $\mathcal{D}(X)$ of diagonal step operators on X and hence identify $\mathcal{L}(X)/\mathcal{S}(X)$ for every closed span X of a transfinite block sequence.

Corollary 5.39. *$\mathcal{L}(X)/\mathcal{S}(X) \cong \mathcal{L}(X,\mathfrak{X}_{\omega_1})/\mathcal{S}(X,\mathfrak{X}_{\omega_1}) \cong J_{T_0}^*(\Gamma_X^{(0)})$ for every $X \hookrightarrow \mathfrak{X}_{\omega_1}$ generated by a transfinite block sequence.*

PROOF. This follows from Lemma 5.18, since $\Lambda(\gamma+1)^{(0)}$ and $\Gamma_X^{(0)}$ are order-isomorphic. $\square$

REMARK 5.40. Note that $\mathcal{L}(X)/\mathcal{S}(X) \cong J_{T_0}^*(\Gamma_X)$ if Γ_X is infinite. To see this, fix a transfinite block sequence $(x_\alpha)_{\alpha<\gamma}$ generating X such that $\gamma \geq \omega^2$. Then $\Gamma_X\setminus\{\max\Gamma_X\}$ and $\Lambda(\gamma+1)^{(0)}\setminus\{\omega\}$ are order-isomorphic.

Theorem 5.41. *Every projection P of $\mathfrak{X}_{\omega_1}$ is of the form $P = P_{I_1}+\cdots+P_{I_n}+S$, where I_i are intervals of ordinals, $I_i < I_{i+1}$ and S is strictly singular.*

PROOF. Suppose that $P:\mathfrak{X}_{\omega_1}\to\mathfrak{X}_{\omega_1}$ is a projection, $P = D_P+S$. Since $P^2 = P$, we obtain that $D_P^2 - D_P$ is also strictly singular and therefore $(\xi_P(\alpha)^2-\xi_P(\alpha))i_{\mathfrak{X}_{[\alpha^-,\alpha)},\mathfrak{X}_{\omega_1}}$ is strictly singular for every successor limit α. This implies that $\xi_P : \Lambda(\omega_1+1)^{(0)} \to \{0,1\}$. And since ξ_P has the continuous extension property, there is no strictly increasing sequence $\{\alpha_n\}_n \subseteq \Lambda(\omega_1+1)^{(0)}$ such that $\xi_P(\alpha_{2n}) = 1$ and $\xi_P(\alpha_{2n+1}) = 0$ for every n. $\square$

Corollary 5.42. *For every $n\in\mathbb{N}$ there is some $m\in\mathbb{N}$ such that for every projection P of $\mathfrak{X}_{\omega_1}$ with $\|P\|\leq n$, P can be written as $P = P_{I_1}+\cdots+P_{I_k}+S$ such that $k\leq m$ and $I_1 << I_2 << \cdots << I_k$, where $A << B$ denotes that the interval $(\sup A, \inf B)$ is infinite.*

PROOF. Fix n, and let $P:\mathfrak{X}_{\omega_1}\to\mathfrak{X}_{\omega_1}$ be a projection such that $\|P\|\leq n$. Let j be the first integer such that $m_{2j} > 2n\boldsymbol{C}$. We claim that $m = n_{2j}$ works. For suppose that $P = P_{I_1}+\cdots+P_{I_k}+S$ with $I_1 << \cdots << I_k$ and $k > n_{2j}$. Fix $\varepsilon > 0$. Find a normalized block sequence $(x_1,y_1,\dots,x_{n_{2j}/2},y_{n_{2j}/2})$ such that

(a) $x_i\in\mathfrak{X}_{I_i}$, $y_i\in\mathfrak{X}_{(\sup I_i,\min I_{i+1})}$ for $1\leq i\leq n_{2j}/2-1$, and $y_{n_{2j}/2} > x_{n_{2j}/2}$,

(b) $(x_1,y_1,\dots,x_{n_{2j}/2},y_{n_{2j}/2})$ is $\boldsymbol{C}$-equivalent to $(v_i)_{i=1}^{n_{2j}}$ and

(c) $\|S|F\|\leq\varepsilon$ where $F = \langle(x_1,,y_1,\dots,x_{n_{2j}/2},y_{n_{2j}/2})\rangle$.

Set $x = x_1-y_1+\cdots+x_{n_{2j}/2}-y_{n_{2j}/2}$. Then,

$$\|x\|\leq\boldsymbol{C}\|\sum_{i=1}^{n_{2j}}(-1)^{i+1}v_i\|_{J_{T_0}}\leq\boldsymbol{C}\|\sum_{i=1}^{n_{2j}}t_i\|_{T_0} = \boldsymbol{C}n_{2j}/m_{2j}, \tag{46}$$

and

$$\|P(x)\|\geq\|\sum_{i=1}^{n_{2j}/2}x_i\|-\varepsilon\geq\|\sum_{i=1}^{n_{2j}/2}v_i\|_{J_{T_0}}-\varepsilon = n_{2j}/2-\varepsilon. \tag{47}$$

(46) and (47) imply that $\|P\|\geq(m_{2j}/2-\varepsilon m_{2j}/n_{2j})/\boldsymbol{C}$. Hence, $\|P\|>n$, a contradiction. $\square$

5.4. **Asymptotically equivalent subspaces and $\mathcal{L}(X,\mathfrak{X}_{\omega_1})$.** Our aim here is to extend the results about operators on subspaces generated by a transfinite block sequence to arbitrary subspace.

Definition 5.43. Let X be a subspace of $\mathfrak{X}_{\omega_1}$. A subset Γ of ω_1+1 is said to be a *critical set* of X if the following hold:

(CS1) Γ is closed of limit ordinals, and $0\in\Gamma$.
(CS2) For all $\gamma\in\Gamma$, $\gamma<\Omega$, $P_{(\gamma,\gamma^+)}|X$ is not strictly singular and for all $\alpha\in(\gamma,\gamma^+)$, $P_{(\gamma,\alpha)}|X$ is strictly singular, where γ^+ is the successor of γ in Γ and $\Omega=\max\Gamma$.
(CS3) $P_{[\Omega,\omega_1)}|X$ is strictly singular (we use $P_\emptyset=0$).

Notice that from the definition it follows easily that if Γ is a critical set of X, then $\max\Gamma=\min\{\gamma\le\omega_1\,:\,P_{[\gamma,\omega_1)}|X$ is strictly singular$\}$.

Proposition 5.44. *For every $X\hookrightarrow\mathfrak{X}_{\omega_1}$ a critical set Γ is uniquely defined, denoted by Γ_X.*

Proof. Fix $X\hookrightarrow\mathfrak{X}_{\omega_1}$. We show first that a critical set X exists. We proceed by induction defining an increasing sequence $(\gamma_\alpha)_{\alpha<\omega_1}$ as follows: We set $\gamma_0=0$. Suppose we have defined $(\gamma_\beta)_{\beta<\alpha}$ satisfying conditions (CS1) and (CS2). If α is a limit ordinal, then we set $\gamma_\alpha=\sup_{\beta<\alpha}\gamma_\beta$. Suppose now that α is a successor ordinal. If $P_{[\gamma_{\alpha^-},\omega_1)}|X$ is strictly singular, then we set $\gamma_\alpha=\gamma_{\alpha^-}$. If not, let

$$\gamma_\alpha=\min\{\gamma\in(\gamma_{\alpha^-},\omega_1)\,:\,P_{[\gamma_{\alpha^-},\gamma)}|X \text{ is not strictly singular}\}.$$

Let us observe that if X is separable, then the sequence $(\gamma_\alpha)_{\alpha<\omega_1}$ is eventually constant and we set $\Gamma_X=\{\gamma_\alpha\}_{\alpha<\omega_1}$. If X is nonseparable, then the sequence $(\gamma_\alpha)_{\alpha<\omega_1}$ is strictly increasing and $\Gamma_X=\{\gamma_\alpha\}_{\alpha<\omega_1}\cup\{\omega_1\}$.

Next we prove the uniqueness of Γ_X. Suppose on the contrary, and fix $\Gamma\neq\Gamma'$ two different critical sets. Set $\gamma=\max(\Gamma\cap\Gamma')$. First notice that $\max\Gamma=\max\Gamma'$. So, either $\gamma^+_\Gamma<\gamma^+_{\Gamma'}$ or $\gamma^+_{\Gamma'}<\gamma^+_\Gamma$. This yields a contradiction using the fact that both Γ and Γ' satisfy (CS2). $\square$

Remark 5.45. 1. The critical set Γ_X provides information concerning the structure of the space X. For example the space X is H.I. if and only if $\Gamma_X=\{0,\Omega_X\}$. Also, two subspaces $X,Y\hookrightarrow\mathfrak{X}_{\omega_1}$ are totally incomparable if and only if $\Gamma_X\cap\Gamma_Y=\{0\}$.
2. For a transfinite block sequence $(x_\alpha)_{\alpha<\gamma}$ its critical set is nothing else but the set introduced from Definition 5.20 (2).

Proposition 5.46. *For every $Y\hookrightarrow X$, the corresponding critical set Γ_Y is a subset of Γ_X.*

Proof. This follows by an easy inductive argument. $\square$

Proposition 5.47. *For every separable $X\hookrightarrow\mathfrak{X}_{\omega_1}$ and for every $\varepsilon>0$ there exist an ordinal $\gamma<\omega_1$, a normalized sequence $(y_\alpha)_{\alpha<\gamma}$ in X and a normalized transfinite block sequence $(z_\alpha)_{\alpha<\gamma}$ such that (a) $\sum_{\alpha<\gamma}\|z_\alpha-x_\alpha\|<\varepsilon$ and (b) $\Gamma_X=\Gamma_Z$ where Z is the closed linear span of $(z_\alpha)_{\alpha<\gamma}$.*

Proof. Use Proposition 5.35, and a standard gliding hump argument. $\square$

Definition 5.48. Let $X,Y\hookrightarrow\mathfrak{X}_{\omega_1}$.

(i) We say that X is *asymptotically finer* than Y, $X\le_a Y$, if and only if $\Gamma_X\subseteq\Gamma_Y$.
(ii) We say that X is *asymptotically equivalent* to Y, $X\equiv_a Y$, if and only if $\Gamma_X=\Gamma_Y$.

It follows easily from the above definition that the relation $\leq_a$ is a quasi ordering in the class of the subspaces of $\mathfrak{X}_{\omega_1}$ which from Proposition 5.46 extends the natural inclusion. Notice also that $\equiv_a$ is an equivalence relation.

We now give two alternative formulation of these notions.

Proposition 5.49. *For $X, Y \hookrightarrow \mathfrak{X}_{\omega_1}$ TFAE:*
(1) *$X \leq_a Y$,*
(2) *if $P_I|X$ is not strictly singular, then $P_I|Y$ is not strictly singular, for every interval $I \subseteq \omega_1$, and*
(3) *$d(S_{X'}, S_Y) = 0$ for every $X' \hookrightarrow X$.*

Proof. Let us observe that for a closed infinite interval I, $P_I|X$ is not strictly singular iff there is some $\gamma^+_{\Gamma_X} \in \Gamma_X$ with $\min \Gamma_X < \gamma^+_{\Gamma_X} \leq \max I$. The inverse direction follows immediately from the definition of the critical sets. So assume now that $P_I|X$ is not strictly singular. Set $\gamma_0 = \max\{\gamma \in \Gamma_X : \gamma \leq \min I\}$. Observe that $\gamma_0 \leq \min I < \Omega_X$, hence $\min \Gamma_X < \gamma^+_{\Gamma_X} \leq \max I$ by the minimality of $\gamma^+_{\Gamma_X}$ (Property (CS2)). It is easy to see that the above observation implies easily the equivalence (1) $\Leftrightarrow$ (2). (1) $\Rightarrow$ (3): Suppose that $X' \hookrightarrow X$. Then by Proposition 5.46 and our assumption, $\Gamma_{X'} \subseteq \Gamma_Y$. By Proposition 5.47, we can find two block sequences $(z_n)_n$ and $(w_n)_n$ in $\mathfrak{X}_{0^+_{\Gamma_{X'}}}$ such that *(a)* $\sup_n \max\operatorname{supp} z_n = \sup_n \max\operatorname{supp} w_n = 0^+_{\Gamma_{X'}}$, and *(b)* $d(S_Z, S'_X) = d(S_W, S_Y) = 0$ where Z and W are the closed linear span of $(z_n)_n$ and $(w_n)_n$ respectively. By Corollary 3.9, $d(Z, W) = 0$ and we are done. (3) $\Rightarrow$ (2): Since for every $X' \hookrightarrow X$, $d(S_{X'}, S_Y) = 0$, we obtain that for every $\varepsilon > 0$, and every $X' \hookrightarrow X$ there exists two basic sequences $(z_n)_n$ and $(w_n)_n$ such that $z_n \in S_{X'}$ and $w_n \in S_Y$ for all n and $\sum_n \|z_n - w_n\| < \varepsilon$. Assume now that $P_I|X$ is not strictly singular. Choose $X' \hookrightarrow X$ such that $P_I|X'$ is an isomorphism. Let $(z_n)_n \subseteq X'$ and $(w_n)_n \subseteq Y$ as above. Then $P_I|W$ is isomorphism and hence $P_I|Y$ is not strictly singular. $\square$

Proposition 5.50. *For $X, Y \hookrightarrow \mathfrak{X}_{\omega_1}$ the following are equivalent:* (1) *$X \equiv_a Y$,* (2) *$P_I|X$ is not strictly singular if and only if $P_I|Y$ is not strictly singular, for every interval $I \subseteq \omega_1$, and* (3) *$d(S_{X'}, S_Y) = d(S_{Y'}, S_X) = 0$ for every $X' \hookrightarrow X$, $Y' \hookrightarrow Y$.* $\square$

Corollary 5.51. *(1) For every $X \hookrightarrow \mathfrak{X}_{\omega_1}$ and every $A \subseteq \Gamma_X$ there is $X_A \hookrightarrow X$ such that $\Gamma_{X_A} = A$.*
(2) For any nonseparable $X, Y \hookrightarrow \mathfrak{X}_{\omega_1}$ there are nonseparable $X_1 \hookrightarrow X$, $Y_1 \hookrightarrow Y$ such that $X_1 \equiv_a Y_1$. $\square$

We shall need the following consequence of a well known result from Lindenstrauss [16].

Lemma 5.52. *Let $Z \hookrightarrow X \hookrightarrow \mathfrak{X}_{\omega_1}$ with Z separable. Then there exist a separable subspace W of X and $\gamma < \omega_1$ such that $Z \hookrightarrow W$ and $P_\gamma|X$ is a projection onto W.* $\square$

Remark 5.53. Notice that for W and X as in the Lemma, Γ_W is an initial part of Γ_X.

Proposition 5.54. *Let X be a subspace of $\mathfrak{X}_{\omega_1}$ and $T : X \to \mathfrak{X}_{\omega_1}$ a bounded operator. Then there exists a unique $D_T \in \mathcal{D}_{\Gamma_X}(\mathfrak{X}_{\omega_1})$ such that (a) $\|D_T\| \leq 2\boldsymbol{C}^2\|T\|$ and (b) $T - D_T|X$ is strictly singular.*

Proof. Fix $X \hookrightarrow \mathfrak{X}_{\omega_1}$ and a bounded operator $T : X \to \mathfrak{X}_{\omega_1}$. First suppose that X is separable. Then we can find a transfinite basic sequence $(y_\alpha)_{\alpha<\gamma} \subseteq X$ and a transfinite block sequence $(z_\alpha)_{\alpha<\gamma}$ of $\mathfrak{X}_{\omega_1}$ such that $\sum_{\alpha<\gamma} \|y_\alpha - z_\alpha\| < 1$ and $X \equiv_a Z$, where Z denotes the closed linear span of $(z_\alpha)_{\alpha<\gamma}$. Consider now $T' : Z \overset{U}{\to} Y \overset{T|Y}{\to} \mathfrak{X}_{\omega_1}$ where Y is the closed linear span of $(y_\alpha)_{\alpha<\gamma}$ and $U : Y \to Z$ is the isomorphism defined by $U(\sum_{\alpha<\gamma} a_\alpha z_\alpha) = \sum_{\alpha<\gamma} a_\alpha y_\alpha$. Notice that $\|U\| \le 2$. Then there is unique $D \in \mathcal{D}(Y)$ such that $T' - D$ is strictly singular, or equivalently there is unique $D_{T'} \in \mathcal{D}_{\Gamma_Z}(\mathfrak{X}_{\omega_1})$ such that $T' - D_{T'}|Z$ is strictly singular. Notice that $\|D_{T'}\| \le \boldsymbol{C}\|D\| \le \boldsymbol{C}^2\|T'\| \le \boldsymbol{C}^2\|U\|\|T\| \le 2\boldsymbol{C}^2\|T\|$. Let us show that $T - D_{T'}$ is strictly singular. Let $X' \hookrightarrow X$ and $\varepsilon > 0$.

Choose $Z' \hookrightarrow Z$ such that $(\Gamma_{Z'} \setminus \{0\}) \cap (\Gamma_{X'} \setminus \{0\}) \neq \emptyset$, $\|U|Z' - i_{Z',\mathfrak{X}_{\omega_1}}\| \le \varepsilon/(4\|T\|)$ and $\|(T' - D_{T'})|Z'\| \le \varepsilon/4$. Pick $z' \in Z'$ and $x' \in X'$ such that $\|z' - x'\| \le \varepsilon/(2(\|D_{T'}\| + \|T\|))$. Then

$$\begin{aligned}\|(T - D_{T'})x'\| \le& \|(T - D_{T'})x' - (T' - D_{T'})z'\| + \|(T' - D_{T'})z'\| \le \|T\|\|x' - Uz'\| + \\ &+ \|D_{T'}\|\|x' - z'\| + \frac{\varepsilon}{4} \le (\|T\| + \|D_{T'}\|)\|x' - z'\| + \frac{\varepsilon}{2} \le \varepsilon. \end{aligned} \tag{48}$$

Now suppose that X is nonseparable. By Lemma 5.52, we can find a sequence $(X_\gamma)_{\gamma<\omega_1}$ of separable complemented subspaces of X such that Γ_{X_γ} is an initial part of Γ_X for every $\gamma < \omega_1$. Now the result for X follows easily from the result for the corresponding $T_\gamma = T|X_\gamma$ and the fact that $D_T \in \mathcal{D}_{\Gamma_X}(\mathfrak{X}_{\omega_1})$ and $D_{T_\gamma} \in \mathcal{D}_{\Gamma_{X_\gamma}}(\mathfrak{X}_{\omega_1})$ are unique. The uniqueness of $D_T \in \mathcal{D}_{\Gamma_X}(\mathfrak{X}_{\omega_1})$ is clear from the analogous result for transfinite block sequences. □

Theorem 5.55. *$\mathcal{L}(X, \mathfrak{X}_{\omega_1}) \cong \mathcal{D}_{\Gamma_X}(\mathfrak{X}_{\omega_1}) \oplus \mathcal{S}(X, \mathfrak{X}_{\omega_1}) \cong J^*_{T_0}(\Gamma^{(0)}_X) \oplus \mathcal{S}(X, \mathfrak{X}_{\omega_1})$ for every $X \hookrightarrow \mathfrak{X}_{\omega_1}$. If in addition Γ_X is infinite, then $\mathcal{L}(X, \mathfrak{X}_{\omega_1}) \cong J^*_{T_0}(\Gamma_X) \oplus \mathcal{S}(X, \mathfrak{X}_{\omega_1})$.*

Proof. Let $H : \mathcal{D}_{\Gamma_X} \to \mathcal{L}(X, \mathfrak{X}_{\omega_1})$ be defined by $D \mapsto D|X$. Assume first that X is separable. It is clear that $\|D|X\| \le \|D\|$. For an appropriate $\varepsilon' > 0$, we can find normalized $(y_\alpha)_{\alpha<\gamma}$ and a normalized block sequence $(z_\alpha)_\alpha$ such that $\Gamma_X = \Gamma_Z$ and $\sum_\alpha \|z_\alpha - y_\alpha\| \le \varepsilon'$ where Z the closed linear span of $(z_\alpha)_{\alpha<\gamma}$. Since by Theorem 5.21 $\|D|Z\| \ge \|D\|/\boldsymbol{C}$, we get that

$$\|D\|/\boldsymbol{C} \le \|D|Z\| \le (1+\varepsilon)\|D|Y\| \le (1+\varepsilon)\|D|X\| = (1+\varepsilon)\|H(D)\|. \tag{49}$$

Hence, H defines an isomorphism. To show that H is an isomorphism when X is nonseparable we use a family $(X_\alpha)_{\alpha<\omega_1}$ of separable complemented subspaces of X defined as in the previous proof. Proposition 5.54 shows that $\mathcal{L}(X, \mathfrak{X}_{\omega_1}) \cong \mathcal{D}_{\Gamma_X}(\mathfrak{X}_{\omega_1}) \oplus \mathcal{S}(X, \mathfrak{X}_{\omega_1})$.

For the later isomorphism see Remark 5.40. □

5.5. **Examples with $\boldsymbol{\mathcal{L}(X)/\mathcal{S}(X) \not\cong \mathcal{L}(X, \mathfrak{X}_{\omega_1})/\mathcal{S}(X, \mathfrak{X}_{\omega_1})}$.** We present a family $\{\mathcal{Z}_\zeta\}_{\zeta<\omega_1}$ of separable subspaces of $\mathfrak{X}_{\omega_1}$ such that each $\mathcal{Z}_\zeta$ is indecomposable but has a ζ-closed direct sum as a subspace.

Definition 5.56. For given $\alpha \le \beta < \omega_1$, let $d_{\alpha,\beta} = e_\alpha + e_\beta$. Given $A = \{\alpha_n\}_n \uparrow, B = \{\beta_n\}_n \uparrow \subseteq \omega_1$ such that $A < B$, let $\mathcal{Z}_{A,B}$ be the closed linear span generated by $\{d_{\alpha_n,\beta_n}\}_n$.

Proposition 5.57. *$\Gamma_{\mathcal{Z}_{A,B}} = \{0, \alpha, \beta\}$ where $\alpha = \sup A$, $\beta = \sup B$.*

Proof. We get the direct inclusion above, since $\mathcal{Z}_{A,B} \subseteq \mathfrak{X}_{A\cup B}$. It remains to show that $P_\alpha|\mathcal{Z}_{A,B}$ and $P_{(\alpha,\beta)}|\mathcal{Z}_{A,B}$ are not strictly singular. We check the case of $P_\alpha|\mathcal{Z}_{A,B}$ since the other is similar. Let $U : \mathfrak{X}_A \to \mathcal{Z}_{A,B}$ be the linear map defined by $e_{\alpha_n} \mapsto d_n$. Since $\lim_n d(Ue_{\alpha_n}, \mathbb{R}e_{\alpha_n}) \geq 1$, we can apply Proposition 5.24, and we can obtain a block sequence $(x_n)_n$ such that $\|Ux_n\| = 1$ and $\|x_n\| < 1/2^n$ for every n. Now $\|P_\alpha Ux_n\| \geq \|Ux_n\| - \|x_n\| \geq 1/2$ for every n. Hence, $P_\alpha|X$ is an isomorphism where X is the closed linear span of the Schauder basic sequence $(Ux_n)_n$. $\square$

Remark 5.58. Note that this shows that $\mathcal{Z}_{A',B'} \equiv_a \mathcal{Z}_{A,B}$ for every infinite $A' \subseteq A$, $B' \subseteq B$.

Proposition 5.59. *Suppose that $T : \mathcal{Z}_{A,B} \to \mathfrak{X}_{\omega_1}$ is bounded and satisfies for every n, m*

$$e^*_{\alpha_n} T d_m = e^*_{\beta_n} T d_m. \tag{50}$$

Then there is some scalar λ such that $T - \lambda i_{\mathcal{Z}_{A,B},\mathfrak{X}_{\omega_1}}$ is strictly singular. Consequently, every bounded operator $T : Z \to Z$ is of the form $T = \lambda Id_Z + S$, where S is strictly singular. Hence, Z is indecomposable.

Proof. Let $T : \mathcal{Z}_{A,B} \to \mathfrak{X}_{A,B}$ be bounded and satisfying (50). Let $d_n = d_{\alpha_n,\beta_n}$ for every n.

Claim. $\lim_{n\to\infty} d(Td_n, \mathbb{R}d_n) = 0$.

Proof of Claim: Condition (50) implies that

$$\max\{\inf_n d(P_\alpha Td_n, \mathbb{R}e_{\alpha_n}), \inf_n d(P_{[\alpha,\omega_1)} Td_n, \mathbb{R}e_{\beta_n})\} > 0. \tag{51}$$

Without loss of generality we may assume that $\inf_n d(P_\alpha Td_n, \mathbb{R}e_{\alpha_n}) > 0$. Applying Proposition 5.24 to $U = P_\alpha T : \langle e_{\alpha_n}\rangle_n \to \mathfrak{X}_{\omega_1}$, we can find $x = \sum_{k\in F} a_k e_{\alpha_k} \in \langle e_{\alpha_n}\rangle_n$ such that $\|x\| < (1/3\|T\|)\|Ux\|$ and $\|\sum_{k\in F} a_k e_{\beta_k}\| \leq (1/3\|T\|)\|Ux\|$. This implies that $\|\sum_{k\in F} a_k d_k\| \leq (2/3\|T\|)\|T(\sum_{k\in F} a_k d_k)\| \leq (2/3)\|\sum_{k\in F} a_k d_k\|$, a contradiction. $\square$

Now for each n, let $\lambda_n \in \mathbb{R}$ realizing $d(Td_n, \mathbb{R}d_n) = \|Td_n - \lambda_n d_n\|$, and choose any accumulation point λ of $(\lambda_n)_n$. Let us show that $S = T - \lambda_{\mathcal{Z}_{A,B}}$ is strictly singular. Fix $\varepsilon > 0$, and let $N \subseteq \mathbb{N}$ be infinite such that $\lambda_n \to_{n\to\infty, n\in N} \lambda$ and $\|T - \lambda i_{\mathcal{Z}_{A',B'}}\| \leq \varepsilon/2$, where $A' = \{\alpha_n\}_{n\in N}$, $B' = \{\beta_n\}_{n\in N}$. Notice that from Remark 5.58 we know also that $\mathcal{Z}_{A',B'} \equiv_a \mathcal{Z}_{A,B}$. So, given any $X \hookrightarrow \mathcal{Z}_{A,B}$, we can find normalized $x \in X$, $y \in \mathcal{Z}_{A',B'}$ with $\|x - y\| \leq \varepsilon/2\|S\|$. Hence, $\|Sx\| \leq \|Sy\| + \|S(x-y)\| \leq \varepsilon$ as desired. $\square$

We generalize the previous ideas and we present a family $\mathcal{Z}_\gamma$ ($\gamma < \omega_1$, γ limit) of infinite dimensional closed subspaces of $\mathfrak{X}_{\omega_1}$ such that for every limit ordinal γ, $\mathcal{Z}_\gamma \equiv_a \mathfrak{X}_\gamma$ and such that $\dim \mathcal{L}(\mathcal{Z}_\gamma)/\mathcal{S}(\mathcal{Z}_\gamma) = 1$. In particular, each $\mathcal{Z}_\gamma$ is an indecomposable space.

Definition 5.60. Fix a limit ordinal $\gamma < \omega_1$. Let $\mathcal{I}_\gamma$ be the family of minimal infinite intervals of γ, i.e., $\mathcal{I}_\gamma = \{[\alpha, \alpha+\omega) : \alpha \text{ is a limit ordinal }, \alpha + \omega \leq \gamma\}$. For each $I \in \mathcal{I}_\gamma$, we choose a partition $\{L^I_J \subseteq I : J \in \mathcal{I}_\gamma\}$ into infinite sets. Notice that since $I = [\alpha, \alpha+\omega)$ for some limit α, all infinite sets L^I_J have order type ω. Now for each $n \in \mathbb{N}$ we consider the vectors $d^{I,J}_n = e_{\alpha_n} + e_{\beta_n}$, where $\{\alpha_n\}_n$ and $\{\beta_n\}_n$ is the increasing enumeration of the sets L^I_J and L^J_I respectively. Finally, let $\mathcal{Z}_\gamma$ be the closed linear span of $(d^{I,J}_n)_{I,J\in\mathcal{I}_\gamma, n\in\mathbb{N}}$.

Theorem 5.61. $\mathcal{L}(\mathcal{Z}_\gamma, \mathfrak{X}_{\omega_1})/\mathcal{S}(\mathcal{Z}_\gamma, \mathfrak{X}_{\omega_1}) \cong J^*_{T_0}(\Lambda(\gamma))$ *and* $\dim \mathcal{L}(\mathcal{Z}_\gamma)/\mathcal{S}(\mathcal{Z}_\gamma) = 1$, *for a limit ordinal γ.*

Proof. Notice that for every limit ordinal α such that $\alpha+\omega \le \gamma$ we have that $d_n^{[\alpha,\alpha+\omega),[\alpha,\alpha+\omega)} = 2e_{\alpha_n} \in \mathcal{Z}_\xi$, where $L_I^I = \{\alpha_n\}_n \uparrow$. This together with the fact that $\mathcal{Z}_\gamma \hookrightarrow \mathfrak{X}_\gamma$ gives that $\mathcal{Z}_\gamma \equiv_a \mathfrak{X}_\gamma$ (i.e, $\Gamma_{\mathcal{Z}_\gamma} = \Lambda(\gamma+1)$) and hence $\mathcal{L}(\mathcal{Z}, \mathfrak{X}_{\omega_1})/\mathcal{S}(\mathcal{Z}, \mathfrak{X}_{\omega_1}) \cong J_{T_0}^*(\Lambda(\gamma+1)^{(0)}) \cong J_{T_0}^*(\Lambda(\gamma))$.

Fix now a bounded operator $T : \mathcal{Z}_\zeta \to \mathcal{Z}_\zeta$. By Proposition 5.54 there is $D \in \mathcal{D}(\mathfrak{X}_\gamma)$ such that $T - D|\mathcal{Z}_\gamma$ is strictly singular. The proof will finish by proving that ξ_T is constant. We observe that given $I < J \in \mathcal{I}_\gamma$, $I = \{\alpha_n\}_n$, $J = \{\beta_n\}_n$ increasing enumeration, we have that $e_{\alpha_n}^* T d_m^{I,J} = e_{\beta_n}^* T d_m^{I,J}$ for every n, m. So from Proposition 5.59 we obtain that for every pair $I < J$ in $\mathcal{I}_\gamma$ there is some $\lambda_{I,J}$ such that $T|\mathcal{Z}_{I,J} - \lambda_{I,J} i_{\mathcal{Z}_{I,J},\mathfrak{X}_{\omega_1}}$ is strictly singular, and this clearly implies that ξ_T is constant. □

Remark 5.62. Notice that it is not possible to improve the previous result to a nonseparable subspace of $\mathfrak{X}_{\omega_1}$, since every nonseparable reflexive space admits non trivial projections [16].

5.6. Further results on Operators.

Corollary 5.63. *No closed linear span X of a transfinite block sequence of $\mathfrak{X}_{\omega_1}$ is isomorphic to finite cartesian power of a Banach space.*

Proof. This is so since $\mathcal{L}(X)$ admits a non trivial linear multiplicative functional. □

Recall the following facts about semi-Fredholm operators (see [13], [17])

Proposition 5.64. *Suppose that $T : X \to Y$ bounded such that TX is closed and $\alpha(T) < \infty$. Then there is some number $\varepsilon(T) > 0$ such that if $S : X \to Y$ is bounded and satisfying that for any $X_1 \hookrightarrow X$ there is some $x \in S_{X_1}$ with $\|S(x)\| < \varepsilon$, then $T + S$ has closed range and $\alpha(T+S) < \infty$.*

Proof. Since $\mathrm{Ker}T$ is finite dimensional, $X = X_1 \oplus \mathrm{Ker}T$. Let $T_1 = T|X_1$. Notice that $T_1|X_1 = TX_1 = TX \hookrightarrow Y$ is closed, and therefore $T_1 : X_1 \to TX_1$ is an isomorphism. Let $\varepsilon = \varepsilon(T) = (1/2)\|T_1^{-1}\|^{-1}$. Fix now S satisfying the condition about ε. Suppose that $T+S$ has $\alpha(T+S) = \infty$. Then, $T_1 + S|X_1$ has infinite dimensional kernel. □

Proposition 5.65. *Suppose that $T : X \to Y$ is semi-Fredholm.*

1. *Then there is some number $\varepsilon = \varepsilon(T) > 0$ such that for all $S : X \to Y$ with $\|S\| < \varepsilon$, $T + S$ is semi-Fredholm and $i(T+S) = i(T)$.*
2. *If $\alpha(T)$ finite, and $S : X \to Y$ is a strictly singular operator, then $T+S$ is semi-Fredholm, $\alpha(T+S)$ is finite and $i(T+S) = i(T)$.* □

In the next results X denotes the closed linear span of a transfinite block sequence $(x_\alpha)_{\alpha<\gamma}$ of $\mathfrak{X}_{\omega_1}$.

Proposition 5.66. *Suppose that $D \in \mathcal{D}(X)$ is such that $\inf\{\xi_D(\alpha) : \alpha \in \Lambda(\gamma+1)^{(0)}\} > 0$. Then D is a Fredholm operator with index 0, and hence it is an onto isomorphism.*

Proof. Let $\widetilde{\xi}_D : \Lambda(\gamma+1) \to \mathbb{R}$ be the unique continuous extension of ξ_D. Notice that the above inequality is equivalent to say that $\widetilde{\xi}_D$ is never zero. In order to show that D is an onto isomorphism it is enough to show that DX is closed. If not, for every n we can find an block sequence $X_n \hookrightarrow \mathfrak{X}_\gamma$ such that $\|D|X_n\| \le 2^{-n}$. Notice that for every n, $D|X_n -$

$\widetilde{\xi}_D(\gamma_n)i_{X_n,\mathfrak{X}_\gamma}$ is strictly singular, where $\gamma_n = \max_n\{\operatorname{supp} x : x \in \langle X_n\rangle\}$. Now for all n, we can find a norm 1 normalized vector $x_n \in X_n$ such that $\|Dx_n - \widetilde{\xi}_D(\gamma_n)x_n\| < 2^{-n}$, and hence $|\widetilde{\xi}_D(\gamma_n)| \le 2^{1-n}$ for every n. Continuity of ξ_D^e implies that there is some limit $\alpha \le \gamma$ such that $\widetilde{\xi}_D(\alpha) = 0$, a contradiction. □

Theorem 5.67. *$T \in \mathcal{L}(X)$ is Fredholm with index $i(T) = 0$ iff it is semi-Fredholm.*

Proof. Suppose that $T : X \to X$ is Semi-Fredholm. Let us take the decomposition $T = D_T + S$. If $\alpha(T)$ is finite, then since S is strictly singular, by Proposition 5.65 (2), D_T is semi-Fredholm with finite dimensional kernel and with the same index as T. This implies that ξ_T is never zero (otherwise, the kernel is infinite dimensional), and hence D_T is indeed 1-1. Since for all $\alpha < \gamma$, $x_\alpha \in D_T\mathfrak{X}_\gamma$, and $D_T\mathfrak{X}_\gamma$ is closed, we get that D_T is an onto isomorphism. Hence T is Fredholm with index 0.

Suppose now that $\beta(T)$ is finite. Let $\varepsilon > 0$ be given by Proposition 5.65 (1), and let $\lambda \in (-\varepsilon, \varepsilon) \setminus \widetilde{\xi}_T(\Lambda(\gamma+1))$. Notice that this is possible since $\widetilde{\xi}_T(\Lambda(\gamma+1))$ is countable by the fact that $\widetilde{\xi}_T : \Lambda(\gamma+1) \to \mathbb{R}$ is continuous. Then $T' = T - \lambda Id_X$ is Semi-Fredholm with the same index as T and $\widetilde{\xi}_{T'}$ is never zero. So, $D_{T'}$ satisfies that $\widetilde{\xi}_{D_{T'}}$ is never zero. By the previous Proposition 5.66, $D_{T'}$ is an isomorphism onto. Hence T' is Fredholm with index 0 and $i(T) = i(T') = 0$. □

Corollary 5.68. *X is not isomorphic to either a proper subspace of it, or to a non trivial quotient.*

Proof. Let $Y \hookrightarrow X$. Suppose first that $T : X \to Y$ is an onto isomorphism. Then the composition $U = i_{Y,X} \circ T : X \to X$ is a semi-Fredholm operator, with $\alpha(T) = 0$. By Theorem 5.67, U is indeed Fredholm with index 0, hence U is onto, i.e, $X = UX = Y$.

Suppose now that $T : X/Y \to X$ is an onto isomorphism. Now the composition $U = T \circ \pi_Y : X \to X$ is semi-Fredholm and onto, where $\pi_Y : X \to X/Y$ is the quotient mapping. Again U has to be Fredholm with index 0, hence U is 1-1, i.e., $Y = \operatorname{Ker} U = \{0\}$. □

Proposition 5.69. $\overline{\langle v_{2n}^*\rangle_n} \cong T_0^*$.

Proof. Fix a sequence of scalars $(b_n)_n$. Let $\sum_n a_n v_n \in J_{T_0}$ be of norm 1 such that

$$\|\sum_n b_n v_{2n}^*\|_{J_{T_0}^*} = (\sum_n b_n v_{2n}^*)(\sum_n a_n v_n) = \sum_n b_n a_{2n}. \tag{52}$$

Since $\|\sum_n a_{2n}t_n\|_{T_0} = \|\sum_n a_{2n}t_{2n}\|_{T_0} \le \|\sum_n a_n v_n\|_{J_{T_0}} = 1$, it follows that $\|\sum_n b_n t_n^*\|_{T_0^*} \ge (\sum_n b_n t_n^*)(\sum_n a_{2n}t_n) = \sum_n b_n a_{2n} = \|\sum_n b_n v_{2n}^*\|_{J_{T_0}^*}$. The other inequality follows from the fact that $(-v_{2n-1} + v_{2n})_n$ is equivalent to $(t_n)_n$ (by Proposition 5.3). □

Proposition 5.70. *There are $X, Y \hookrightarrow \mathfrak{X}_{\omega_1}$ such that $\mathcal{L}(X)/\mathcal{S}(X) \cong \mathcal{L}(Y)/\mathcal{S}(Y) \cong J_{T_0}^*$ and such that $\mathcal{L}(X,Y)/\mathcal{S}(X,Y) \cong T_0^*$.*

Proof. Let $X = \mathfrak{X}_{\omega^2}$, and let $Y = \mathfrak{X}_A$, where $A = \bigcup_n[\omega(2n), \omega(2n+1))$. The result follows from the fact that if $T : X \to Y$ is a bounded operator, then necessarily $\xi_{T'}(\omega(2n)) = 0$, for $T' = i_{Y,X} \circ T = D + S$. And $\mathcal{L}(X,Y)/\mathcal{S}(X,Y) \cong \{\xi \in J_{T_0}^* : \xi(v_{2n+1}) = 0\} = \overline{\langle v_{2n}^*\rangle_n} \cong T_0^*$. □

Proposition 5.71. *Every $T \in \mathcal{L}(\mathfrak{X}_{\omega_1})$ is of the form $T = \lambda Id + R$, where R has separable range.*

PROOF. We know that $T = D_T + S$ with S strictly singular. Corollary 5.25 shows that $S \in c_0(\omega_1)$. Since $\xi_T : \Lambda(\omega_1)^{(0)} \to \mathbb{R}$ has the continuous extension property, there is some $\lambda \in \mathbb{R}$ and some ordinal $\alpha < \omega_1$ such that $\xi_T(\beta) = \lambda$ for every $\beta \in \Lambda(\omega_1)^{(0)} \cap [\alpha, \omega_1)$. Hence $D_T - \lambda Id$ has separable range. So $T - \lambda Id = (D_T - \lambda Id) + S$ has separable range, as desired. $\square$

Nonseparable spaces with this property of operators have been constructed before in [23], [24] and [28]. These constructions however give no control on separable subspaces.

The following theorems summarize our results for the structure of $\mathfrak{X}_{\omega_1}$, its subspaces and the spaces of operators.

Theorem 5.72. *There exists a reflexive space $\mathfrak{X}_{\omega_1}$ with a transfinite basis $(e_\alpha)_{\alpha<\omega_1}$ such that*
(1) It does not contain an unconditional basic sequence.
(2) It is arbitrarily distortable.
(3) $\mathfrak{X}_I$ and $\mathfrak{X}_J$ are totally incomparable for disjoint infinite intervals I and J.
(4) It is ω_1 hereditarily indecomposable (i.e., for every nonseparable $X, Y \hookrightarrow \mathfrak{X}_{\omega_1}$, $dist(S_X, S_Y) = 0$).
(5) Every subspace $X \hookrightarrow \mathfrak{X}_{\omega_1}$ generated by a transfinite block sequence is, neither isomorphic to a proper subspace, nor to a non-trivial of its quotients.

To each infinite dimensional subspace X of $\mathfrak{X}_{\omega_1}$ we assign a closed subset Γ_X of ω_1, called the critical set of X. The following theorem describes the interference of X and Γ_X.

Theorem 5.73. *For X, Y subspaces of $\mathfrak{X}_{\omega_1}$ the following holds*
(1) If $Y \hookrightarrow X$ then $\Gamma_Y \subset \Gamma_X$.
(2) The subspaces X, Y are totally incomparable iff $\Gamma_X \cap \Gamma_Y = \{0\}$.
(3) The subspace X is hereditarily indecomposable iff $\#\Gamma_X = 2$.
(4) For every subspace X of $\mathfrak{X}_{\omega_1}$ there exists Y generated by a block sequence $(y_\alpha)_{\alpha<\gamma}$ such that $\Gamma_X = \Gamma_Y$.

Finally the structure of the spaces of operators is described by the next theorem. Recall that $\Gamma_X^{(0)}$ is the set of isolated ordinals of Γ_X.

Theorem 5.74. *(1) For every $X \hookrightarrow \mathfrak{X}_{\omega_1}$, $\mathcal{L}(X, \mathfrak{X}_{\omega_1}) \cong \mathcal{D}_{\Gamma_X}(\mathfrak{X}_{\omega_1}) \oplus \mathcal{S}(X, \mathfrak{X}_{\omega_1}) \cong J^*_{T_0}(\Gamma_X^{(0)}) \oplus \mathcal{S}(X, \mathfrak{X}_{\omega_1})$. If in addition Γ_X is infinite, then $\mathcal{L}(X, \mathfrak{X}_{\omega_1}) \cong J^*_{T_0}(\Gamma_X) \oplus \mathcal{S}(X, \mathfrak{X}_{\omega_1})$.*
(2) For every $X \hookrightarrow \mathfrak{X}_{\omega_1}$ generated by a transfinite block sequence, $\mathcal{L}(X) \cong J^*_{T_0}(\Gamma_X^{(0)}) \oplus \mathcal{S}(X)$. If in addition Γ_X is infinite, $\mathcal{L}(X) \cong J^*_{T_0}(\Gamma_X) \oplus \mathcal{S}(X)$.
(3) For every $\gamma < \omega_1$ there exists a subspace Y_γ of $\mathfrak{X}_{\omega_1}$ such that $\mathcal{L}(Y_\gamma) \cong \langle \mathrm{Id}_{Y_\gamma} \rangle \oplus \mathcal{S}(Y_\gamma)$ and $\mathcal{L}(Y_\gamma, \mathfrak{X}_{\omega_1}) \cong J^*_{T_0}(\gamma) \oplus \mathcal{S}(Y_\gamma, \mathfrak{X}_{\omega_1})$.

6. Universal and smooth ϱ-functions

In this section we present two new properties which a $\varrho-$function can have. In this and in the subsequent section we show how these properties of ϱ influence the corresponding space $\mathfrak{X}_{\omega_1}$ based on σ_ρ.

6.1. **The construction of a universal ϱ-function.** In this subsection we show how the construction of the ϱ-function of [26] can be adjusted in order to give us a function $\varrho : [\omega_1]^2 \to \omega$ with the following properties:

1. $\varrho(\alpha,\gamma) \leq \max\{\varrho(\alpha,\beta),\varrho(\beta,\gamma)\}$ for all $\alpha<\beta<\gamma<\omega_1$.
2. $\varrho(\alpha,\beta) \leq \max\{\varrho(\alpha,\gamma),\varrho(\beta,\gamma)\}$ for all $\alpha<\beta<\gamma<\omega_1$.
3. $\{\alpha<\beta : \varrho(\alpha,\beta)\leq n\}$ is finite for all $\beta<\omega_1$ and $n\in\mathbb{N}$.
4. (ω_1,ϱ) is universal.

To describe what we mean by "(ω_1,ϱ) is universal" we need some more definitions.

Definition 6.1. A finite ϱ-model is a model of the form $(M,<,\varrho_M,p_M)$ where M is a set, $<$ is a total ordering on M, p_M is an integer and $\varrho_M : [M]^2 \to p_M$ is a function with properties 1. and 2. listed above. We also assume that there exist $x<y$ in M such that $\varrho_M(x,y)=p_M$.

Definition 6.2. Suppose that $\varrho : [\lambda]^2 \to \omega$ satisfies 1., 2., and 3. above. For $M\subseteq\lambda$, let $p_M=\max\{\varrho(\alpha,\beta) : \alpha,\beta\in M\}$. Such an M is ϱ-closed if

$$M=\{\alpha<\lambda : \exists\beta\in M\,(\alpha\leq\beta\,\&\varrho(\alpha,\beta)\leq p_M\,)\}.$$

We use the convention of $\varrho(\alpha,\alpha)=0$ for all α. Note that for a ϱ-closed subset M of λ, $(M,<,\varrho|[M]^2,p_M)$ is an example of a ϱ-model. Note also that an initial part M_0 of a ϱ-closed set M is a ϱ-closed set and that its integer p_{M_0} might be smaller than p_M. Similarly, an initial part of a ϱ-model is also a ϱ-model with a possibly smaller integer p_M.

Definition 6.3. Two ϱ-models $(M_1,<_1,\varrho_1,p_1)$ and $(M_2,<_2,\varrho_2,p_2)$ are isomorphic if $p_1=p_2$ and if there is order-isomorphism $\pi : (M_1,<_1)\to(M_2,<_2)$ such that $\varrho_1(a,b)=\varrho_2(\pi(a),\pi(b))$ for all $a,b\in M_1$.

Definition 6.4. A function $\varrho : [\lambda]^2\to\omega$ defined on some limit ordinal $\lambda\leq\omega_1$ and satisfying 1., 2. and 3. is said to be *universal* if for every finite ϱ-model $(M,<,\varrho_M,p_M)$, every ϱ-closed subset M_0 of λ such that $(M_0,<,\varrho|[M_0]^2,p_{M_0})$ is isomorphic to an initial segment of $(M,<,\varrho_M,p_M)$, and every ordinal δ such that $\delta+\omega\leq\lambda$, there is a ϱ-closed subset M_1 of $\delta+\omega$ such that

5. $(M_1,<,\varrho|[M_1]^2,p_{M_1})\cong(M,<,\varrho_M,p_M)$
6. M_0 is an initial segment of M_1.
7. $M_1\setminus M_0\subseteq[\delta,\delta+\omega)$.

The existence of a universal $\varrho : [\omega_1]^2\to\omega$ is established by recursively constructing an increasing sequence $\varrho_\lambda : [\lambda]^2\to\omega$ $(\lambda\in\Lambda)$. Let $\varrho_0=\emptyset$, and suppose $\varrho_\lambda : [\lambda]^2\to\omega$ has been determined for some countable limit ordinal λ. Let C be a subset of λ of order-type ω such that $\lambda=\sup C$. Define

$$\varrho_{\lambda+\omega}(\alpha,\lambda)=\max\{\#(C\cap\alpha),\varrho_\lambda(\alpha,\min(C\setminus\alpha)),\varrho_\lambda(\xi,\alpha) : \xi\in C\cap\alpha\}.$$

It can be checked (see e.g. [26]) that this defines a function $\varrho_{\lambda+\omega} : [\lambda+1]^2\to\omega$ satisfying the conditions 1., 2. and 3. Starting with this extension of ϱ_λ and the assumption that ϱ_λ is universal we build extensions $\varrho_{\lambda+\omega} : [\delta]^2\to\omega$ $(\lambda+1\leq\delta<\lambda+\omega)$ in such a way that at a given stage δ we take care about a particular instance of universality of $\varrho_{\lambda+\omega}$. Thus, modulo some book keeping device, it suffices to show how one deals with the following task: We have already defined an extension $\varrho_{\lambda+\omega} : [\delta]^2 \to \omega$, we are given a finite ϱ-model $(M, <, \varrho_M, p_M)$ and a ϱ-closed subset M of δ such that $(M_0, <, \varrho_{\lambda+\omega}|[m_0]^2, p_{M_0})$ is isomorphic to a proper initial segment of $(M, <, \varrho_M, p_M)$. Let $l = \#M - \#M_0$ and extend $\varrho_{\lambda+\omega}$ from $[\delta]^2$ to $[\delta+l]^2$ as follows: First of all define $\varrho_{\lambda+\omega}$ on

$$[M_0 \cup [\delta, \delta+l)]^2 \setminus [M_0]^2$$

in such a way that we have the isomorphism

$$(M_0 \cup [\delta, \delta+l), <, \varrho_{\lambda+\omega}, p_M) \cong (M, <, \varrho_M, p_M).$$

Thus, it remains to define $\varrho(\alpha, \gamma)$ for $\alpha \in \delta \setminus M_0$ and $\gamma \in (\delta, \delta+l)$. If $\alpha < \delta$ and $\alpha > \max M_0$, then set

$$\varrho(\alpha, \gamma) = \max\{p_M + 1, \varrho(\alpha, \delta-1), \varrho(\xi, \alpha) \,:\, \xi \in M_0\}.$$

If $\alpha \leq \max M_0$, then set

$$\varrho(\alpha, \gamma) = \max\{\varrho(\alpha, \min(M_0 \setminus \alpha)), \varrho(\xi, \alpha) \,:\, \xi \in M_0 \cap \alpha\}.$$

It remains to show that $\varrho_{\lambda+\omega}|[\delta+l]^2$ satisfies the properties 1. and 2. So let $\alpha < \beta < \gamma < \delta + l$ be given. We simplify the notation by writing $\alpha\beta$ instead of $\varrho(\alpha, \beta)$, $\xi\eta \vee \alpha\beta$ instead $\max\{\varrho(\xi, \eta), \varrho(\alpha, \beta)\}$, and δ^- in place of $\delta - 1$.

Case 1. $\alpha < \delta \leq \beta < \gamma < \delta + l$. If $\alpha \in M_0$, then properties 1. and 2. for $\alpha < \beta < \gamma$ follow from the fact that in the definitions of $\alpha\beta$, $\beta\gamma$ and $\alpha\gamma$ we have copied the ϱ-model $(M, <, \varrho_M, p_M)$ which satisfies 1. and 2.

If $\alpha \notin M_0$, then in both the case $\alpha > \max M_0$ and $\alpha \leq \max M_0$ we conclude that $\alpha\beta = \alpha\gamma$, so 1. and 2. for $\alpha < \beta < \gamma$ follow immediately.

Case 2. $\alpha < \beta < \delta < \gamma < \delta + l$. *Subcase 2.1.* $\max M_0 < \delta < \beta$. Consider first the inequality $\alpha\gamma \leq \alpha\beta \vee \beta\gamma$. The quantities $p_M + 1$ and $\xi\alpha$ $(\xi \in M)$ from the definition of $\alpha\gamma$ are all present in the definition of $\beta\gamma$, so it remains only to show that the quantity $\alpha\delta^-$ is bounded by $\alpha\beta \vee \beta\gamma$. Applying 1. for $\varrho_{\lambda+\omega}|[\delta]^2$ we get

$$\alpha\delta^- \leq \alpha\beta \vee \beta\delta^-,$$

and so we are done as $\beta\delta^-$ shows up in the definition of $\beta\gamma$. Consider now the inequality $\alpha\beta \leq \alpha\gamma \vee \beta\gamma$. Applying 2. for $\varrho_{\lambda+\omega}|[\delta]^2$ to the triple $\alpha < \beta < \delta^-$ we get

$$\alpha\beta \leq \alpha\delta^- \vee \beta\delta^-,$$

so we are done also in this case since the quantity on the right-hand side is bounded by $\alpha\gamma \vee \beta\gamma$.

Subcase 2.2. $\alpha \leq \max M_0 < \beta < \delta$. Consider first the subcase when $\alpha \in M_0$. To see that $\alpha\gamma \leq \alpha\beta \vee \beta\gamma$ observe that $\alpha\gamma \leq p_M < p_M + 1 \leq \beta\gamma$. To see that $\alpha\beta \leq \alpha\gamma \vee \beta\gamma$ observe that $\alpha\beta$ appear as a quantity int the definition of $\beta\gamma$. Let us consider the case $\alpha \notin M_0$ and let

$$\alpha' = \min(M_0 \setminus \alpha).$$

The quantity $\alpha\alpha'$ from the definition of $\alpha\gamma$ is bounded by $\alpha\beta \vee \beta\gamma$ since by 2. for $\varrho_{\lambda+\omega}|[\delta]^2$ we have that

$$\alpha\alpha' \leq \alpha\beta \vee \alpha'\beta,$$

and $\alpha'\beta$ appears in the definition of $\beta\gamma$. Since the quantities $\xi\alpha$ ($\xi \in M_0 \cap \alpha$) appear also in the definition of $\beta\gamma$, this establishes the inequality $\alpha\gamma \leq \alpha\beta \vee \beta\gamma$. Consider now the inequality $\alpha\beta \leq \alpha\gamma \vee \beta\gamma$. Apply 1. of $\varrho_{\lambda+\omega}|[\delta]^2$ to $\alpha < \alpha' < \beta$ and get

$$\alpha\beta \leq \alpha\alpha' \vee \alpha'\beta,$$

and this finishes the proof since $\alpha\alpha' \leq \alpha\gamma$ and $\alpha'\beta \leq \beta\gamma$.

Subcase 2.3. $\alpha < \beta \max M_0$. If $\alpha, \beta \in M_0$, then the inequalities 1. and 2. for $\alpha < \beta < \gamma$ follow from the fact that in the definitions of $\alpha\beta$, $\beta\gamma$ and $\alpha\gamma$ we copied the ϱ-model $(M, <, \varrho_M, p_M)$.

Subcase 2.3.1 $\alpha \in M_0$ and $\beta \notin M_0$. Consider the inequality $\alpha\gamma \leq \alpha\beta \vee \beta\gamma$. This follows from the fact that

$$\alpha\gamma \leq p_M < \beta\beta', \text{ where } \beta' = \min(M_0 \setminus \beta)$$

and the fact that in the definition of $\beta\gamma$ the quantity $\beta\beta'$ appears. The inequality $\alpha\beta \leq \alpha\gamma \vee \beta\gamma$ in this subcase follows from the fact that the quantity $\alpha\beta$ appears in the definition of $\beta\gamma$.

Subcase 2.3.2 $\alpha \notin M_0$ and $\beta \in M_0$. Consider the inequality $\alpha\gamma \leq \alpha\beta \vee \beta\gamma$. Let

$$\alpha' = \min(M_0 \setminus \alpha).$$

We need to bound the quantities $\alpha\alpha'$ and $\xi\alpha$ ($\xi \in M_0 \cap \alpha$) by $\alpha\beta \vee \beta\gamma$. Apply 2. of $\varrho_{\lambda+\omega}|[\delta]^2$ to $\alpha < \alpha' \leq \beta$ and get

$$\alpha\alpha' \leq \alpha\beta \vee \alpha'\beta.$$

Since $\alpha'\beta \leq p_M$ and $\alpha\alpha' > p_M$ we conclude that $\alpha\alpha' \leq \alpha\beta$ as required. Similarly note that

$$\xi\alpha \leq \xi\beta \vee \alpha\beta = \alpha\beta,$$

since $\xi\beta \leq p_M$ while $\alpha\beta > p_M$. It remains to check the inequality $\alpha\beta \leq \alpha\gamma \vee \beta\gamma$ in this subcase. As before note that

$$\alpha\beta \leq \alpha\alpha' \vee \alpha'\beta,$$

and that $\alpha\beta > \alpha'\beta$ since $\alpha'\beta \leq p_M$ while $\alpha\beta > p_M$. It follows that $\alpha\beta \leq \alpha\alpha' \leq \alpha\gamma$ as required.

Subcase 2.3.3 $\alpha\beta \notin M_0$ (and $\alpha, \beta \leq \max M_0$). So in this subcase both quantities $\alpha\gamma$ and $\beta\gamma$ are defined according to the second definition. Let

$$\alpha' = \min(M_0 \setminus \alpha) \text{ and } \beta' = \min(M_0 \setminus \beta).$$

Note that $\alpha' \leq \beta'$. We first check the inequality $\alpha\gamma \leq \alpha\beta \vee \beta\gamma$. Consider first the quantity $\alpha\alpha'$ that appears in the definition of $\alpha\gamma$. If $\alpha' = \beta'$, then

$$\alpha\alpha' \leq \alpha\beta \vee \beta\beta' \leq \alpha\beta \vee \beta\gamma,$$

as $\beta\beta'$ appears in the definition of $\beta\gamma$. Suppose that $\alpha' < \beta'$ i.e., that $\alpha' < \beta$. Then

$$\alpha\alpha' \leq \alpha\beta \vee \alpha' \leq \alpha\beta \vee \beta\gamma$$

as $\alpha'\beta$ appears as a quantity in the definition of $\beta\gamma$. Consider the quantity $\xi\alpha$ for $\xi \in M_0 \cap \alpha$. Note that

$$\xi\alpha \leq \alpha\beta \vee \xi\beta \leq \alpha\beta \vee \beta\gamma,$$

as $\xi\beta$ appears in the definition of $\beta\gamma$. It remains to check the inequality $\alpha\beta \le \alpha\gamma \vee \beta\gamma$ in this subcase. If $\alpha' = \beta'$, then we get that

$$\alpha\beta \le \alpha\beta' \vee \beta\beta' \le \alpha\gamma \vee \beta\gamma,$$

as the quantity $\alpha\beta' = \alpha\alpha'$ appears in $\alpha\gamma$ while $\beta\beta'$ appears in $\beta\gamma$. If $\alpha' < \beta'$ i.e., $\alpha' < \beta \le \beta'$, then we get that

$$\alpha\beta \le \alpha\alpha' \vee \alpha'\beta \le \alpha\gamma \vee \beta\gamma,$$

since $\alpha\alpha'$ appears in $\alpha\gamma$ and $\alpha'\beta$ appears in $\beta\gamma$. This finishes checking that the extension $\varrho_{\lambda+\omega}|[\delta+l]^2$ satisfies the conditions 1. 2. and 3. Note that

$$\varrho(\alpha,\gamma) > p_M \text{ for all } \alpha \in \delta \setminus M_0 \text{ and } \gamma \in [\delta, \delta+l),$$

we conclude that the set $M_0 \cup [\delta, \delta+l)$ is $\varrho_{\lambda+\omega}$-closed. So the extension $\varrho_{\lambda+\omega}|[\delta,\delta+l]^2$ has a set

$$M_1 = M_0 \cup [\delta, \delta+1) \subseteq \delta + l,$$

which is $\varrho_{\lambda+\omega}$-closed while the corresponding model $(M_1, <, \varrho_{\lambda+\omega}|[M_1]^2, p_M)$ is isomorphic to the given ϱ-model $(M, <, \varrho_M, p_M)$. This finishes the recursive construction of a universal ϱ-function $\varrho : [\omega_1]^2 \to \omega$. The reader is referred to [26] for more on ϱ-functions and their uses. Some of the applications need the following *unboundedness property*, stronger than 3.

3'. For every $n < \omega$ and infinite $M \subseteq \omega_1$ there exist $\alpha < \beta$ in M such that $\varrho(\alpha, \beta) > n$.

As there is no reason to suspect that the universal $\varrho : [\omega_1]^2 \to \omega$ just produced satisfies 3'. we offer the following derived function $\bar{\varrho} : [\omega_1]^2 \to \omega$:

$$\bar{\varrho}(\alpha,\beta) = \max\{\varrho(\alpha,\beta), \#(\{\xi \le \alpha \,:\, \varrho(\xi,\alpha) \le \varrho(\alpha,\beta)\})\}.$$

It may be checked that $\bar{\varrho}$ has the properties 1., 2. and 3'.

6.2. **A smooth ϱ-function.** We construct a ϱ-function such that the corresponding coding σ_ϱ yields that $\mathfrak{X}_\xi$ has a Schauder basis for every $\xi < \omega_1$. These bases will be a reordering of the transfinite basis $(e_\alpha)_{\alpha<\xi}$ in order type ω.

For a given ϱ-function and an ordinal $\alpha < \omega_1$, let $F_n^\alpha = \{\beta < \alpha \,:\, \varrho(\beta, \alpha) \le n\}$, which are n-closed.

Definition 6.5. A ϱ-function is called *smooth* if for every limit ordinal $\lambda < \omega_1$, the numerical sequence $(\#F_n^\lambda/n)_n$ is bounded.

Proposition 6.6. *There exists a smooth ϱ-function.*

Proof. Let us show that such smooth ϱ-function exists. The definition will be again inductive, i.e., for each limit ordinal λ we are going to define $\varrho_\lambda : [\lambda]^2 \to \omega$. Suppose we have defined ϱ_λ, and set

$$\varrho_{\lambda+\omega}(\alpha,\lambda) = \max\{g_\lambda(\#(C_\lambda \cap \alpha)), \varrho_\lambda(\alpha, \min(C_\lambda \setminus \alpha)), \varrho_\lambda(\xi,\alpha) \,:\, \xi \in C_\lambda \cap \alpha\} \tag{53}$$

where $g_\lambda : \mathbb{N} \to \mathbb{N}$ is increasing and C_λ is cofinal in λ and they are defined as follows: For a given $\alpha < \lambda$, let $i(\alpha) = \#C_\lambda \cap \alpha$. Suppose we have constructed ϱ_λ such that for all limit $\gamma < \lambda$ with the smooth property $\lim_{n\to\infty} \#F_n^\gamma/n = 0$. There are two cases.

(a) Suppose that $\lambda = \gamma + \omega$ is a successor limit. For each integer i, let $g_\lambda(i) = 2^i$ and $C_\lambda = \{\gamma + n\}_n$. Notice that for $\alpha < \lambda$, if $\varrho(\alpha, \lambda) \le n$, then either $\alpha < \gamma$ and $\alpha \in F_n^\gamma$ or if $\alpha = \gamma + l$, then since $i(\alpha) = \#C_\lambda \cap \alpha = l + 1$, we have that $l \le \log_2(n)$. So, $\#F_n^\lambda \le \#F_n^\gamma + 1 + \log_2 n$, which certainly implies that $\#F_n^\lambda/n \to_n 0$.
(b) Suppose that λ is a limit of limit ordinals. Let $C_\lambda = \{\lambda_n\}_n \subseteq \lambda$ cofinal in λ, with each λ_n a limit ordinal. Let $(n_i)_i$ be a strictly increasing sequence of integers such that

$$\#F_n^{\lambda_0}/n + \cdots + \#F_n^{\lambda_i}/n \le 2^{-i} \tag{54}$$

for every i and every $n \ge n_k$. Let $g_\lambda(i) = n_i$ for all i. Fix $\varepsilon > 0$, and let j be such that $1/2^j \le \varepsilon$. We show that for all $n \ge n_j$, $\#F_n^\lambda/n \le \varepsilon$: Fix $n \ge n_j$, and let i_0 be the maximal integer such that $n_j \le n_{i_0} \le n$. Notice that then

$$F_n^\lambda \subseteq \{\alpha < \lambda \,:\, i(\alpha) \le i_0 \text{ and } \alpha \in F_n^{\lambda_{i(\alpha)}}\} = F_n^{\lambda_0} \cup \cdots \cup F_n^{\lambda_{i_0}}. \tag{55}$$

So, $\#F^\lambda/n \le \#F_n^{\lambda_0}/n + \cdots + \#F_n^{\lambda_{i_0}}/n \le 2^{-i_0} \le 2^{-j} \le \varepsilon$. □

Lemma 6.7. *Let G be a p-closed set, and let $\phi = (1/w(\Phi))\sum_{i=1}^d \phi_i \in K$ be of odd weight $w(\phi) < p$, and such that $\operatorname{supp}\phi \cap G \neq \emptyset$. If defined, set*

$$d_1 = \max\{i \in [1, d] \,:\, w(\Phi_i) < p\} \textit{ and } d_0 = \max\{i < d_1 \,:\, \operatorname{supp}\Phi_i \cap G \neq \emptyset\},$$

Then
(1) $(1/w(\phi))\sum_{i=1}^{d_0} \Phi_i|[0,\alpha]$ has support contained in G, where $\alpha = \max(\operatorname{supp}\Phi_{d_0} \cap G)$.
(2) $\operatorname{supp}\phi_i \cap G = \emptyset$ for every $d_0 < i < d_1$.
(3) $w(\phi_i) \ge p$ for every $d_1 < i \le d$.

Proof. If d_1 is not well defined, then for all $k \le d$, $w(\phi_k) \ge p$. If d_1 is well defined, but d_0 is not, then for all $k < d_1$ we have that $\operatorname{supp}\phi_k \cap G = \emptyset$. Suppose that both are well defined.

Finally, since $p \ge w(\phi_{d_1}) \ge \max\{p_\varrho(\bigcup_{i=1}^{d_0} \operatorname{supp}\phi_i), w(\phi_i)\}$ and $\overline{\bigcup_{i=1}^{d_0} \operatorname{supp}\phi_i \cap (\alpha+1)}^p \subseteq G$ (G is p-closed), it follows that the support of $(1/w(\phi))\sum_{i=1}^{d_0} \Phi_i|[0,\alpha]$ is included in G. □

Lemma 6.8. *Let $G \subseteq \omega_1$ be p-closed. Then for all $\phi \in K_{\omega_1}$ there are some f_0 and f_1 such that*

1. $\operatorname{supp} f_0, \operatorname{supp} f_1 \subseteq G$, $f_0 + f_1 = \phi|G$
2. $\|f_0\|_\infty \le 1/p$,
3. $f_1 \in 2K_{\omega_1}(G)$, *where $K_{\omega_1}(G)$ is the subset of K_{ω_1} consisting on the functionals ϕ with support contained in G.*

Proof. Let $(\phi_t)_{t\in\mathcal{T}}$ be a tree-analysis of ϕ. Let

$$\mathcal{T}_0 = \{t \in \mathcal{T} \,:\, \text{ there is some } u \preceq t \text{ with } w(\phi_u) \ge p\} \text{ and } \mathcal{T}_1 = \mathcal{T} \setminus \mathcal{T}_0.$$

Notice that $\mathcal{T}_1$ is a downwards closed subtree of $\mathcal{T}$, and hence for a given $t \in \mathcal{T}_1$, the set S_t^1 of immediate successor of t in $\mathcal{T}_1$ is exactly equal to $S_t^1 = S_t \cap \mathcal{T}_1$. If $\mathcal{T}_1 = \emptyset$, then $\phi_0 = \phi$ has to be of type I and $w(\phi_0) \ge p$. In this case, let $f_0 = \phi$ and $f_1 = 0$, that clearly satisfies what we want. Suppose now that $\mathcal{T}_1 \neq \emptyset$. We are going to find for all $t \in \mathcal{T}_1$, f_0^t, f_1^t such that

1. $\operatorname{supp} f_0^t, \operatorname{supp} f_1^t \subseteq G$, $f_0^t + f_1^t = \phi|G$
2. $\|f_0^t\|_\infty \le 1/p$,
3. $f_1^t \in 2K_{\omega_1}$.

It is clear that the pair $f_0 = f_0^0$, $f_1 = f_1^0$ satisfies our requirements. The proof goes by downwards induction over $t \in \mathcal{T}_1$ on the tree $\mathcal{T}_1$. Suppose that $t \in \mathcal{T}_1$ is a terminal node of $\mathcal{T}_1$.
(1) If t is terminal node of $\mathcal{T}$, then we set $f_0^t = 0$ and $f_1^t = \phi_t$ if $\operatorname{supp}\phi_t \subseteq G$, and $f_0^t = f_1^t = 0$ otherwise.
(2) If t is not terminal node of $\mathcal{T}$, then this means that for all $s \in S_t$, ϕ_s is of type I and $w(\phi_s) \geq p$. Set $f_0^t = \phi_t|G$, $f_1^t = 0$.

Suppose now that $t \in \mathcal{T}_1$ is not terminal in $\mathcal{T}_1$. Clearly this implies that t is not terminal in $\mathcal{T}$. There are three cases: *Case 1.* ϕ_t is of type II, $\phi_t = \sum_{s\in S_t} r_s\phi_s$. Then we set

$$f_0^t = \sum_{s\in S_t^1} r_s f_0^s + \sum_{s\in S_1\setminus\mathcal{T}_1} r_s\phi_s|G \text{ and } f_1^t = \sum_{s\in S_t^1} r_s f_1^s.$$

Since for $s \in S_t$, $s \notin \mathcal{T}_1$ iff ϕ_s is of type I and $w(\phi_s) \geq p$, this gives that $\|f_0^t\|_\infty \leq 1/p$. The rest our inductive promises for f_0^t and f_1^t are clearly satisfied.
Case 2. ϕ_t is of type I, and $w(\phi_t)$ is even. We set

$$f_0^t = \frac{1}{w(\phi_t)}\sum_{s\in S_t^1} f_0^s + \frac{1}{w(\phi_t)}\sum_{s\in S_t\setminus\mathcal{T}_1} \phi_s|G \text{ and } f_1^t = \frac{1}{w(\phi_t)}\sum_{s\in S_t^1} f_1^s.$$

The condition $\|f_0^t\|_\infty \leq 1/p$ is satisfied by the same reason as in the previous case.
Case 3. ϕ_t is of type I, and $w(\phi_t)$ is odd, $\phi_t = (1/w(\phi_t))\sum_{i=1}^d \phi_{s_i}$, where $\{s_1,\dots,s_d\} = S_t$. Find $d_0 < d_1 \leq d$ as in the previous Lemma 6.7. If d_1 is not well defined, this implies that $w(\phi_s) \geq p$ for every $s \in S_t$. Then $S_t^1 = \emptyset$ and we set $f_0^t = \phi_t|G$ and $f_1^t = 0$. Suppose that d_1 is well defined but d_0 is not. This means that $\operatorname{supp}\phi_k \cap G = \emptyset$ for every $k < d_1$. Then we set

$$f_0^t = \frac{1}{w(\phi_t)}\left(f_0^{s_{d_1}} + \sum_{i=d_1+1}^{d}\phi_{s_i}|G\right) \text{ and } f_1^t = \frac{1}{w(\phi_t)} f_1^{s_{\lambda_{\phi,\phi'}}}.$$

Suppose now that both d_0 and d_1 are well defined, then we set

$$f_0^t = \frac{1}{w(\phi_t)}\left(f_0^{s_{d_1}} + \sum_{i=d_1+1}^{d}\phi_{s_i}|G\right) \text{ and } f_1^t = \frac{1}{w(\phi_t)}\left(\sum_{i=1}^{d_0}\phi_i|[0,\alpha]\right) + \frac{1}{w(\phi_t)} f_1^{s_{\lambda_{\phi,\phi'}}},$$

where $\alpha = \max(\operatorname{supp}\phi_{d_0} \cap G)$. Notice that $1/w(\phi_t)(\sum_{i=1}^{d_0}\phi_i|[0,\alpha]) \in K_{\omega_1}(G)$. Therefore, using the induction hypothesis we conclude that $f_1^t \in 2K_{\omega_1}$. $\square$

Lemma 6.9. *Assume that $\mathfrak{X}_{\omega_1}$ is built upon a smooth ϱ-function and fix a limit ordinal $\lambda < \omega_1$. Then the projections $(P_{F_n^\lambda})_n$ are uniformly bounded by $2 + D_\lambda$, where $D_\lambda = \sup_n \#F_n^\lambda/n < \infty$.*

Proof. Fix a limit ordinal λ, let $x \in \mathfrak{X}_\lambda$ be of norm 1, and $\phi \in K_{\omega_1}$.Take the decomposition $\phi = f_0 + f_1$ from the previous Lemma 6.8 applied to the n-closed set F_n^λ. Then,

$$|\phi P_{F_n^\lambda}x| = |\langle f_0, P_{F_n^\lambda}x\rangle + \langle f_1, P_{F_n^\lambda}x\rangle| \leq \frac{\#F_n^\lambda}{n} + |\langle f_1, P_{F_n^\lambda}x\rangle| \leq D_\lambda + |\langle f_1, P_{F_n^\lambda}x\rangle|. \tag{56}$$

Now using that $f_1 \in 2K_{\omega_1}(F_n^\lambda)$, we can write $f_1 = \sum_i \lambda_i\phi_i$, $\sum_i \lambda_i \leq 2$, $\lambda_i \geq 0$, and $\phi_i \in K_{\omega_1}(F_n^\lambda)$. Therefore, $\langle \phi_i, P_{F_n^\lambda}x\rangle = \langle \phi_i, x\rangle \leq 1$. So, $|\langle f_1, P_{F_n^\lambda}x\rangle| \leq 2$, and we are done. $\square$

Let $Q_n^\lambda = P_{F_n^\lambda}$. Notice that $Q_n^\lambda Q_m^\lambda = Q_{\min\{n,m\}}^\lambda$.

Theorem 6.10. *For every $x \in \mathfrak{X}_\lambda$, $\lim_{n\to\infty} Q_n^\lambda(x) = x$.*

Proof. Let us show that for all limit $\beta \leq \lambda$ and all $x \in \mathfrak{X}_\beta$, $\lim_{n\to\infty} P_{F_n^\lambda} x = x$. The proof is by the induction over the set of limit ordinals $\leq \lambda$. Fix $x \in \mathfrak{X}_\beta$.
(a) $\beta = \omega$. We know that $\lim_{n\to\infty} P_n x = x$. Fix $\varepsilon > 0$, and let n_0 be such that for all $n \geq n_0$, $\|x - P_n x\| \leq \varepsilon/(3 + D_\lambda)$. Let $n_1 \geq n_0$ be such that for all $n \geq n_1$, $[0, n_0] \subseteq F_n^\lambda$. Hence $\|x - P_{F_n^\lambda} x\| \leq \|x - P_{n_0} x\| + \|P_{F_n^\lambda}(x - P_{n_0}x)\| \leq (1 + \|P_{F_n^\lambda}\|)\|x - P_{n_0}x\| \leq \varepsilon$ for every $n \geq n_1$.
(b) $\beta = \gamma + \omega$. Then, $x = y + z$, where $y \in \mathfrak{X}_\gamma$, and $z \in \mathfrak{X}_{[\gamma,\gamma+\omega)}$. By the induction hypothesis, $\lim_{n\to\infty} P_{F_n^\lambda} y = y$. Now use the projections $(P_{[\gamma,\gamma+n)})_n$ to approximate z and follow the ideas of the case $\beta = \omega$.
(c) β is limit of limit ordinals. Fix a strictly increasing sequence $(\beta_n)_n$ with limit β, and let $x_n = P_{\beta_n} x$. We know that $\lim_{n\to\infty} x_n = x$. Fix $\varepsilon > 0$, and let n_0 be such that $\|x - x_{n_0}\| \leq \varepsilon/2(3 + D_\lambda)$. Let $n_1 \geq n_0$ be such that $\|x_{n_0} - P_{F_n^\lambda} x_{n_0}\| \leq \varepsilon/2$ for all $n \geq n_1$, that we know that it is possible by the induction hypothesis since $x_{n_0} \in \mathfrak{X}_{\beta_{n_0}}$. Then for all $n \geq n_1$,

$$\begin{aligned}\|x - P_{F_n^\lambda} x_n\| \leq & \|x - x_{n_0}\| + \|x_{n_0} - P_{F_n^\lambda} x_{n_0}\| + \|P_{F_n^\lambda}\|\|x - x_{n_0}\| \leq \\ \leq & (3 + D_\lambda)\|x - x_{n_0}\| + \|x_{n_0} - P_{F_n^\lambda} x_{n_0}\| \leq \varepsilon.\end{aligned}$$

□

Corollary 6.11. *The space $\mathfrak{X}_\alpha$ has a Schauder basis for every ordinal $\alpha < \omega_1$. Moreover, for every $\alpha < \omega_1$ there exists a reordering $(e_{\beta_n})_n$ of $(e_\beta)_{\beta<\alpha}$ such that $(e_{\beta_n})_n$ is a Schauder basis of the space $\mathfrak{X}_\alpha$.*

Proof. It is enough to show the result for a limit ordinal λ. By the previous Theorem, the projections $(Q_n^\lambda)_n$ define a finite dimensional Schauder decomposition of $\mathfrak{X}_\lambda$. Notice that the natural ordering $<_\lambda$ on λ defined by

$$\alpha <_\lambda \beta \text{ iff } \begin{cases} \varrho(\alpha,\lambda) < \varrho(\beta,\lambda) & \text{or} \\ \varrho(\alpha,\lambda) = \varrho(\beta,\lambda) & \text{and } \alpha < \beta \end{cases}$$

has order type ω. Let $\{\lambda_n\}_n$ be an enumeration of $(\lambda, <_\lambda)$ in order type ω, and let us show that $(x_n = e_{\lambda_n})_n$ is a basis of $\mathfrak{X}_\lambda$: Let $(R_n)_n$ be the projections $R_n : \mathfrak{X}_\lambda \to \mathfrak{X}_\lambda$ associated to $(x_n)_n$. For a given k, let $n_k = \varrho(\lambda_k, \lambda)$. Then, $R_k = Q_{n_k-1}^\lambda + P_{\lambda_k} \circ (Q_{n_k}^\lambda - Q_{n_k-1}^\lambda)$. This clearly shows that $(x_n)_n$ is a Schauder basis of $\mathfrak{X}_\lambda$. □

Remark 6.12. It is unclear whether there is a variation on ϱ such that some of the resulting spaces $\mathfrak{X}_\lambda$ $(\lambda < \omega_1)$ do not admit Schauder basis.

7. Universality of ϱ and nearly subsymmetric bases

Throughout this section we assume that the coding σ_ϱ is based on an universal ϱ function discussed in the previous section.

Remark 7.1. For the sequel we need a slight modification of the definition of special sequences. More precisely, we assume that for each $(\phi_1, w_1, p_1, \dots, \phi_{n_{2j+1}}, w_{n_{2j+1}}, p_{n_{2j+1}})$ every p_i satisfies the additional property that for all $l \leq i$, ϕ_l admits a tree-analysis with supports in the set

$$G_i = \overline{\bigcup_{k=1}^{l} \operatorname{supp} \phi_k}^{p_i} .$$

Note that the definition of the special functionals and the fact that K_{ω_1} is rationally closed does not allow one to conclude that every functional $\phi \in K_{\omega_1}$ admits a tree-analysis $(\phi_t)_{t\in\mathcal{T}}$ such that for every $t \in \mathcal{T}$, $\operatorname{supp} \phi_t \subseteq \operatorname{supp} \phi$. However there will always be large enough p such that $\overline{\operatorname{supp} \phi}^p$ contains a tree-analysis of ϕ. This follows from the fact that there is a tree-analysis $(\phi_t)_{t\in\mathcal{T}}$ of ϕ such that $\max \bigcup_{t\in\mathcal{T}} \operatorname{supp} \phi_t = \max \phi$.

Definition 7.2. For a given p, and a subset $G \subseteq \omega_1$ let

$$K_p(G) = \{\phi \in K \,:\, \phi \text{ has some tree-analysis } (\phi_t)_{t\in\mathcal{T}} \text{ such that } \forall t \in \mathcal{T} \operatorname{supp} \phi_t \subseteq G \text{ and}$$
$$\text{if } \phi_t \text{ has type I, then } w(\phi_t) < p\}.$$

We will call such tree-analysis $\mathcal{F} = (\phi_t)_{t\in\mathcal{T}}$ of $\phi \in K_p(G)$ a *(p, G)-tree-analysis* of ϕ. Notice that if $\mathcal{F} =$ is a (p, G)-tree-analysis of ϕ, then for all interval E, $(\phi_t|E)_{t\in\mathcal{T}}$ is a (p, G)-tree-analysis of $\phi|E$.

Proposition 7.3. *Suppose that $G, G' \subseteq \omega_1$ are p-complete, and (ϱ-)isomorphic (see Definition 6.3). Then the unique order preserving mapping $\pi : G \to G'$ defines a bijection*

$$\tilde{\pi} : K_p(G) \to K_p(G')$$

such that for every $\alpha \in G$ $\tilde{\pi}(e^*_\alpha) = e^*_\beta$, preserves (p, G)-tree-analysis in $K_p(G)$ and weights.

Proof. The proof is an easy use of downwards induction over a (p, G)-tree-analysis. □

Using the properties of our new coding σ_ϱ we can improve Lemma 6.8 as follows.

Lemma 7.4. *Let $G \subseteq \omega_1$ be p-closed. Then for all $\phi \in K_{\omega_1}$ there are some f_0 and f_1 such that*
1. $\operatorname{supp} f_0, \operatorname{supp} f_1 \subseteq G$, $f_0 + f_1 = \phi|G$,
2. $\|f_0\|_\infty \leq 1/p$, *and*
3. $f_1 \in 2K_p(G)$.

Proof. The Proof follows exactly the same steps than the proof of Lemma 6.8 with the exception that the inductive premise 3. $\operatorname{supp} f_1^t \in 2K_{\omega_1}(G)$ now is replaced by $f_1^t \in 2K_p(G)$. To check that one can find the corresponding decomposition when one deals with the case of odd weight, we notice that the premise 3. will be fulfilled since the new coding σ_ϱ will guarantee that in Lemma 6.7, the corresponding $(1/w(\phi)) \sum_{i=1}^{d_0} \phi_i|[0, \alpha] \in K_p(G)$. □

Definition 7.5. A transfinite basis $(e_\alpha)_{\alpha<\gamma}$ is said to be *C-nearly subsymmetric* if for every $\varepsilon > 0$ and for every family of finite successive subsets $\{F_i\}_{i=1}^d$ of γ and every family $\{I_i\}_{i=1}^d$ of successive infinite intervals there exists $\{G_i\}_{i=1}^d$ with $G_i \subseteq I_i$, $\#G_i = \#F_i$ such that the natural isomorphism $T : \langle (e_\alpha)_{\alpha\in\cup_{i=1}^d F_i}\rangle \to \langle (e_\beta)_{\beta\in\cup_{i=1}^d G_i}\rangle$ satisfies $\|T\| \cdot \|T^{-1}\| \leq C + \varepsilon$.

The purpose of this section is to prove the following result.

Theorem 7.6. *The transfinite basis $(e_\alpha)_{\alpha<\omega_1}$ is 4-nearly subsymmetric.*

Proof. We want to show that for every sequence of finite sets $F_1 < F_2 < \cdots < F_n$, infinite intervals $I_1 \leq I_2 \leq \cdots \leq I_n$ (with possible repetitions) and $\varepsilon > 0$, there is some $G_1 < G_2 < \cdots < G_n$ such that

(a) $G_i \subseteq I_i$, $i = 1, \dots, n$,

(b) $\#F_i = \#G_i$, $i = 1, \dots, n$,

(c) The natural isomorphism T between $\mathfrak{X}_F$ and $\mathfrak{X}_G$ satisfies that $\|T\|, \|T^{-1}\| \leq 2 + \varepsilon$, where $\mathfrak{X}_F = \langle e_\alpha \rangle_{\alpha \in F}$ and $\mathfrak{X}_G = \langle e_\alpha \rangle_{\alpha \in G}$ and T is defined for $\alpha \in F$ and $\beta \in G$ such that $T(e_\alpha) = e_\beta$ satisfies that $\alpha \mapsto \beta$ is order preserving and onto, and $F = \bigcup_{i=1}^n F_i$ and $G = \bigcup_{i=1}^n G_i$.

Let $p \geq \max\{p_\varrho(\bigcup_{i=1}^n F_i), \#F/\varepsilon\}$, and let $\tilde{F} = \overline{F}^p$. For each $i = 1, \dots, n$, let $\alpha_i = \max F_i + 1$, and let $F_i' = G \cap \alpha_i$. Notice that

(1) $F_i \subseteq F_i'$ for every $i = 1, \dots, n$,

(2) each F_i' is a p-closed set for $i = 1, \dots, n$, and

(3) F_i' is an initial segment of F_j' for $i \leq j \leq n$.

By the universality of ϱ, there is some $G_1' \subseteq I_1$ which is ϱ-isomorphic to F_1'. Since $(F_2', <, \varrho|[F_2']^2, p)$ and $(F_1,', <, \varrho|[F_1']^2, p)$ are ϱ-models, F_1' is an initial segment of F_2', and $(F_1', <, \varrho|[F_1']^2, p) \cong (G_1', <, \varrho|[G_1']^2, p)$, the universality of ϱ gives a set $H_2 \subseteq I_2 \setminus G_1'$ such that $G_2' = G_1' \cup H_2$ satisfies that

(1) G_2' is p-closed, and

(2)$(G_2', <, \varrho|[G_2']^2, p) \cong (F_2', <, \varrho|[F_2']^2, p)$.

And so on. At the end we get n many ϱ models $(G_i', <, \varrho|[G_i']^2, p)$ for $i = 1, \dots, n$ such that

(1) For $i \leq j \leq n$, G_i' is an initial segment of G_j',

(2) $G_i' \setminus G_{i-1}' \subseteq I_i$, for $i = 2, \dots, n$, and $G_1' \subseteq I_1$,

(3) $(G_i', <, \varrho|[G_i']^2, p) \cong (F_i', <, \varrho|[F_i']^2, p)$ for $i = 1, \dots, n$.

Therefore $\tilde{G} = G_n'$ satisfies that $(\tilde{G}, <, \varrho|[G']^2, p) \cong (\tilde{F}, <, \varrho|[F']^2, p)$. Let π be the isomorphism between them, and for each $i = 1, \dots, n$, let $G_i = \pi F_i$. Let us show that $T : \mathfrak{X}_F \to \mathfrak{X}_G$ satisfies what we wanted: Fix one vector $x = \sum_{i \in F} \lambda_i e_i$ such that $\|x\| = 1$. Take $\phi \in K$ such that $\phi x = 1$. Then take the decomposition of $\phi = f_0 + f_1$ as in previous Lemma 7.4. Using Proposition 7.3, we can take a copy g_1 of f_1 in $2K_p(\tilde{G})$. Since $1 = \phi x = |f_0 x + f_1 x| \leq |f_0 x| + N/p < |g_1 T x| + \varepsilon$, $|g_1 T x| > 1 - \varepsilon$. This implies that there is some $\psi \in K_p(\tilde{G})$ such that $|\psi T x| > (1 - \varepsilon)/2$. So, $\|Tx\| \geq (1 - \varepsilon)/2$.

Now suppose that $\|Tx\| > 2 + \varepsilon$. Then, let $\phi \in K$ be such that $\phi T x > 2 + \varepsilon$. Take the decomposition $\phi = g_0 + g_1$ as in the previous Lemma 7.4, now in $K_p(\tilde{G})$. This implies that $g_1 T x > 2$, and hence, there is some $\psi \in K_p(\tilde{G})$ such that $\psi T x > 1$. Hence, the copy ϕ of ψ in $K_{\tilde{F}}(p)$ is such that $\phi x = \psi T x > 1$, a contradiction. So, $\|T\| \leq 2 + \varepsilon$ and $\|T^{-1}\| \leq 2/(1 - \varepsilon) \leq 2 + \varepsilon$. $\square$

Definition 7.7. Recall the following (modified) notion from [19]. Let X be a Banach space with a Schauder basis $(u_n)_n$, fix $n \in \mathbb{N}$ and $C \geq 1$. A finite n-dimensional space E with a basis $(e_i)_{i=1}^n$ is called a C-*asymptotic space* of X iff

$$\sup_{X_1} \inf_{x_1 \in S(X_1)} \sup_{X_2} \dots \inf_{x_n \in S(X_n)} d_b(\langle x_1, \dots, x_n \rangle, E) \leq C, \tag{57}$$

where $d_b = \|T\| \cdot \|T^{-1}\|$ for $T : \langle x_1, \dots, x_n \rangle \to E$ to be the natural isomorphism defined by $T(x_i) = e_i$, and X_i runs over all tail subspaces, i.e, $X_i = \overline{\langle u_i \rangle_{i>k}}$ for some k. A space Y with a monotone basis $(y_n)_n$ is called a *C-asymptotic version* of X iff for every n, $\langle y_i \rangle_{i=1}^n$ is an asymptotic space of X.

Corollary 7.8. *There exists a family $\{X_\gamma\}_{\gamma<\omega_1}$ of reflexive totally incomparable hereditarily indecomposable spaces with Schauder bases such that X_γ is an asymptotic version of $X_{\gamma'}$ for every $\gamma, \gamma' < \omega_1$.* $\square$

Definition 7.9. Two transfinite basis $(x_\alpha)_{\alpha<\gamma_0}$ of X_0 and $(x_\alpha)_{\alpha<\gamma_1}$ of X_1 are called *finitely equivalent* iff there is some constant $C > 0$ such that for all finite set $F_0 \subseteq \gamma_0$ there is some finite set $F_1 \subseteq \gamma_1$ with the same cardinality such that $(x_\alpha)_{\alpha \in F_0}$ and $(y_\alpha)_{\alpha \in F_1}$ are C-equivalent.

REMARK 7.10. There are finitely equivalent subspaces of $\mathfrak{X}_{\omega_1}$ which are incomparable.

REMARK 7.11. Using the fact that ϱ is universal it can be shown that for any bounded $T : \mathfrak{X}_{\omega_1} \to \mathfrak{X}_{\omega_1}$, $\|D_T\| \leq 4\|T\|$. The proof goes as follows. We assume that $\|T\| = 1$. Fix a normalized finitely supported vector x; let $x = x_1 + \dots + x_n$ be its decomposition in $\mathfrak{X}_{\omega_1}$, and $\varepsilon > 0$. Let $F_i = \operatorname{supp} x_i$ for each $i = 1, \dots, n$, and consider infinite intervals $I_1 \leq I_2 \leq \dots \leq I_n$ of ω_1 such that $\|D_T(y) - T(y)\| \leq \varepsilon\|y\|$ for every $y \in \mathfrak{X}_{I_1 \cup \dots \cup I_n}$. By Theorem 7.6 we can find for every $i = 1, \dots, n$ $G_i \subseteq I_i$ such that $\#G_i = \#F_i$ and the order isomorphism between $F = \bigcup_{i=1}^n F_i$ and $G = \bigcup_{i=1}^n G_i$ defines an isomorphism H between $\langle e_\alpha \rangle_{\alpha \in F}$ and $\langle e_\alpha \rangle_{\alpha \in G}$ with $\|H\|, \|H^{-1}\| \leq 2+\varepsilon$. Set $y = F(x)$ and then $\|y\| \leq 2 + \varepsilon$ and $\|(T - D_T)(y) \leq \varepsilon\|$. Since $H(D_T(x)) = D_T(y)$ we have that $\|D_T(x)\| \leq (2 + \varepsilon)\|D_T(y)\|$. So,

$$\|D_T(x)\| \leq (2+\varepsilon)\|D_T(y)\| \leq (2+\varepsilon)(\|D_T(y) - T(y)\| + \|Ty\|) \leq (2+\varepsilon)\varepsilon + (2+\varepsilon)^2\|T\|. \quad (58)$$

8. TREE-ANALYSIS OF FUNCTIONALS: BASIC INEQUALITY AND FINITE INTERVAL REPRESENTABILITY OF J_{T_0}

The goal of this section is to prove the basic inequality (Lemma 4.4) and show the finite interval representability of the James-like space J_{T_0} in $\mathfrak{X}_{\omega_1}$ (Theorem 5.9). Reaching these two goals involve similar sort of problems and for this reason we introduce a general theory applicable to both cases and hopefully to many other cases to come.

8.1. **General theory.** The theory deals with a block sequence of vectors $(x_k)_{k=1}^n$, a sequence of scalars $(b_k)_{k=1}^n$, and a functional $f \in K_{\omega_1}$, and tries to estimate $|f(\sum_{k=1}^n b_k x_k)|$ in terms of $|g(\sum_{k=1}^n b_k e_k)|$ for an appropriately chosen functional g of an auxiliary Tsirelson-like space X with basis $(e_i)_i$. The natural approach is to start with a tree-analysis $(f_t)_{t\in\mathcal{T}}$ of f, and try to replace the functional f_t at each node $t \in \mathcal{T}$ by a functional g_t in the norming set of the auxiliary space, and in doing this try to copy, as much as possible, the given tree-analysis $(f_t)_{t\in\mathcal{T}}$. Not all nodes $t \in \mathcal{T}$ have the same importance in this process. It turns out that the crucial replacements $f_t \mapsto g_t$ are made for t belonging to some sets $\mathcal{A} \subseteq \mathcal{T}$ such that $(f_t)_{t\in\mathcal{A}}$ is in some sense responsible for the estimation of the action of the whole functional f on each of the vectors x_k. These are the *maximal antichains* of $\mathcal{T}$ defined below. Observe that some of the replacements $f_t \mapsto g_t$ are necessary before this procedure has a chance to work. Suppose for example the replacements are made in an auxiliary mixed Tsirelson space X where a particular $(m_{j_0}^{-1}, n_{j_0})$-operation is not allowed. Then, every time we find a node $t \in \mathcal{T}$ such that the corresponding f_t has weight $w(f_t) = m_{j_0}$ the replacement g_t has to be something avoiding this operation, i.e., we cannot put the combination $g_t = (1/w(f_t)) \sum_{s \in S_t} g_s$. These sorts of nodes are the ones that we call “catchers” below, because their own tree-analysis $(f_s)_{s \succeq t}$ cannot be taken into account.

8.1.1. *Antichains and arrays of antichains.* Recall that every $f \in K_{\omega_1}$ has a tree-analysis $(f_t)_{t \in \mathcal{T}}$ such that: For every $t \in \mathcal{T}$ (a) if $u \succeq t$, then $\operatorname{ran} f_u \subseteq \operatorname{ran} f_t$, and (b) if f_t is of type I, then $f_t = (1/w(f_t)) \sum_{s \in S_t} f_s$.

Recall that $A \subseteq \mathcal{T}$ is called an *antichain* if for every $t \neq t' \in A$, neither $t \preceq t'$ nor $t' \preceq t$. Given $t, t' \in \mathcal{T}$, we define $t \wedge t' = \max\{v \in \mathcal{T} \, : \, v \preceq t, t'\}$. Notice that $\mathcal{A} \subseteq \mathcal{T}$ is an antichain iff $t \wedge t' \precneqq t, t'$ for every $t \neq t' \in \mathcal{T}$.

Definition 8.1. Fix a tree-analysis $(f_t)_{t \in \mathcal{T}}$ of f as above. Given a finitely supported vector x, a set $\mathcal{A} \subseteq \mathcal{T}$ is called a *regular antichain* for x and $(f_t)_{t \in \mathcal{T}}$ if

(a.1) for every $t \in \mathcal{A}$, f_t is not of type II,
(a.2) $f_{t_1 \wedge t_2}$ is of type II for every $t_1 \neq t_2 \in \mathcal{A}$, and
(a.3) $\operatorname{ran} f_t \cap \operatorname{ran} x \neq \emptyset$, for every $t \in \mathcal{A}$.

$\mathcal{A}$ is a *maximal antichain* for x if in addition $\mathcal{A}$ satisfies
(a.4) for every $t \in \mathcal{T}$ if $\operatorname{supp} f_t \cap \operatorname{ran} x \neq \emptyset$, then there is some $u \in \mathcal{A}$ comparable with t.

Let $(x_k)_{k=1}^n$ be a block sequence, and let $\boldsymbol{\mathcal{A}} = (\mathcal{A}_k)_{k=1}^n$ be such that each $\mathcal{A}_k$ is a regular antichain for the vector x_k and the tree-analysis $(f_t)_{t \in \mathcal{T}}$. For a given $t \in \mathcal{T}$, we define

$$D_t^{\boldsymbol{\mathcal{A}}} = \bigcup_{u \succeq t} \{k \in [1, n] \, : \, u \in \mathcal{A}_k\}, \; E_t^{\boldsymbol{\mathcal{A}}} = D_t^{\boldsymbol{\mathcal{A}}} \setminus (\bigcup_{s \in S_t} D_s^{\boldsymbol{\mathcal{A}}}).$$

Whenever there is no possible confusion we simply write D_t and E_t to denote $D_t^{\boldsymbol{\mathcal{A}}}$ and $E_t^{\boldsymbol{\mathcal{A}}}$ respectively.

$\boldsymbol{\mathcal{A}} = (\mathcal{A}_k)_{k=1}^n$ is called a *(maximal) regular array* for $(x_k)_{k=1}^n$ and $(f_t)_{t \in \mathcal{T}}$ if each $\mathcal{A}_k$ is a (maximal) regular antichain for x_k and $(f_t)_{t \in \mathcal{T}}$, and in addition
(a.5) for every $t \in \bigcup_k \mathcal{A}_k$ such that f_t is of type I, either t is a *catcher*, i.e., $D_s = \emptyset$ for every $s \in S_t$, or for every $k \in E_t$, t is a *splitter* of x_k, i.e., for every $k \in E_t$ there are at least $s_1 \neq s_2 \in \mathcal{S}_t$ such that $\operatorname{ran} f_{s_i} \cap \operatorname{ran} x_k \neq \emptyset$.

We denote by $\mathrm{S}(\boldsymbol{\mathcal{A}})$ and $\mathrm{C}(\boldsymbol{\mathcal{A}})$ the set of splitter nodes and catcher nodes of $\boldsymbol{\mathcal{A}}$, respectively. Notice that if $t_i \in \mathcal{A}_{k_i}$ $(i = 1, 2)$ are catcher nodes, then they are incomparable, and that $\mathcal{A}_k = \mathrm{S}(\boldsymbol{\mathcal{A}}) \cup \mathrm{C}(\boldsymbol{\mathcal{A}})$.

Note that if no f_t $(t \in \mathcal{T})$ is of type II then $\#\mathcal{A}_k \leq 1$ for all k, and so the tree-analysis below becomes much simpler.

Definition 8.2. *(The functor $\mathcal{A}(x, \mathrm{C})$.)* Given a block vector x and $\mathrm{C} \subseteq \mathcal{T}$ consisting of nodes of type I, let $\mathcal{A}(x, \mathrm{C})$ be the set of nodes $t \in \mathcal{T}$ such that

(A.1) f_t is not of type II,
(A.2) $\operatorname{ran} f_t \cap \operatorname{ran} x \neq \emptyset$,
(A.3) for every $s \preceq t$ if $s \in S_u$ and f_u is of type I, then for every $s' \in S_u \setminus \{s\}$, $\operatorname{ran} f_{s'} \cap \operatorname{ran} x = \emptyset$,

(A.4) if f_t is of type I and $t \notin \mathrm{C}$, then t is a splitter of x.
(A.5) for every $u \precneqq t$, $u \notin \mathrm{C}$.

Proposition 8.3. *$\mathcal{A} = \mathcal{A}(x, \mathrm{C})$ is a maximal regular antichain such that $\{t \in \mathcal{A} \setminus C : f_t$ of type I$\} \subseteq \mathrm{S}(\mathcal{A})$. Moreover, if $(x_k)_{k=1}^n$ is a block sequence, then the corresponding $\boldsymbol{\mathcal{A}} = (\mathcal{A}(x_k, \mathrm{C}))_{k=1}^n$ is a maximal regular array such that*
(a) *$\{t \in \bigcup_k \mathcal{A}_k \setminus C : f_t$ of type I$\} \subseteq \mathrm{S}(\boldsymbol{\mathcal{A}})$, and*
(b) *$\mathrm{C} \subseteq \mathrm{C}(\boldsymbol{\mathcal{A}})$ and for every $t \in \mathrm{C}$, E_t is an interval of integers.*

Proof. Fix $t \neq t' \in \mathcal{A}_k$. $f_{t \wedge t'}$ being of type II follows from the facts that if $u \precneqq t$, then $u \notin \mathrm{C}$, by *(A.5)*, hence if f_u is of type I, then *(A.3)* implies that u is not splitter of x. We show the maximality of $\mathcal{A}$: Fix $t \in \mathcal{T}$ such that $\operatorname{supp} f_t \cap \operatorname{ran} x \neq \emptyset$. Let $t_0 \succeq t$ be such that f_{t_0} is of type 0 and $\operatorname{supp} f_{t_0} \subseteq \operatorname{ran} w_k$, and set $b = [0, t_0] = \{v \in \mathcal{T} : v \preceq t_0\}$ which is a $\preceq$-well ordered set and $t \in b$. We distinguish two cases: Suppose first that $b \cap \mathrm{C} = \emptyset$. Let $u_0 = \min\{u \in b : v$ satisfies *(A.1)*, *(A.4)*$\}$. Notice that u_0 exists since t_0 satisfies *(A.1)* and *(A.4)*. The minimality of u_0 shows that u_0 satisfies *(A.3)*, hence $u_0 \in \mathcal{A}$. Suppose now that $b \cap \mathrm{C} \neq \emptyset$, and set $v_0 = \min b \cap \mathrm{C}$. It is not difficult to show that $u_0 = \max\{u \preceq v_0 : u$ satisfies *(A.1)*, *(A.4)*$\}$ is in $\mathcal{A}$ (notice that v_0 satisfies *(A.1)* and *(A.4)*, hence u_0 is well defined.)

Repeating this procedure for each vector in a given a block sequence $(x_k)_{k=1}^n$, one gets that the array $(\mathcal{A}(x_k, \mathrm{C}))_{k=1}^n$ is maximal and regular. Finally suppose that $t \in \mathrm{C}$ and suppose that $k_1 < k_2 < k_3$ with $k_1, k_3 \in E_t$. It is routine to check that t satisfies *(A1)-(A.5)* for x_{k_2}, hence it follows that $k_2 \in E_t$. □

Proposition 8.4. *Suppose that $\boldsymbol{\mathcal{A}} = (\mathcal{A}_k)_{k=1}^n$ is a regular array for a block sequence $(x_k)_{k=1}^n$ and $(f_t)_{t \in \mathcal{T}}$. Then:*
(b.0) If $t \in \mathcal{A}_k$ is a splitter or if f_t is of type 0, then $\operatorname{supp} f_t \cap \operatorname{ran} x_k \neq \emptyset$.
(b.1) If f_t is of type I, then $\{D_s\}_{s \in S_t} \cup \{\{k\} : k \in E_t\}$ is a block family, and if t is a splitter, then $\# E_t \leq \# S_t - 1$.

Suppose that in addition $\boldsymbol{\mathcal{A}} = (\mathcal{A}_k)_{k=1}^n$ is maximal for $(x_k)_{k=1}^n$.
(b.2) Let $t \in \mathcal{A}_k$, $u \precneqq s \preceq t$ with f_u of type I, and $s \in S_t$. Then for every $s' \in S_u \setminus \{s\}$ $\operatorname{ran} f_{s'} \cap \operatorname{ran} x_k = \emptyset$.
(b.3) Suppose that f_t is of type II, $k \in D_t$ and $s \in S_t$. If $\operatorname{supp} f_t \cap \operatorname{ran} x_k \neq \emptyset$, then $k \in D_s$.

Proof. *(b.0)*: If f_t is of type 0, the conclusion is clear. If t is a splitter, let $s_1 \neq s_2 \in S_t$ be such that $f_{s_1} < f_{s_2}$ and $\operatorname{ran} f_{s_1} \cap \operatorname{ran} x_k, \operatorname{ran} f_{s_2} \cap \operatorname{ran} x_k \neq \emptyset$. Then $\max \operatorname{supp} f_{s_1} \in \operatorname{ran} x_k$.

(b.1): For the first part, if t is a catcher, there is nothing to prove, so we assume t is a splitter. First we show that $\{D_s\}_{s \in S_t} \cup \{\{k\} : k \in E_t\}$ is a disjoint family. If $k \in E_t \cap D_s$ for some $s \in S_t$, then there is some $u \succeq s$ with $u \in \mathcal{A}_k$. But $t \in \mathcal{A}_k$ and $t \precneqq u$, a contradiction. Suppose that $k \in D_s \cap D_{s'}$ with $s \neq s' \in S_t$. Then there are $u, u' \in \mathcal{A}_k$ such that $u \succeq s$, $u' \succeq s'$. Hence $u \wedge u' = t$ but f_t is of type I, contradicting *(a.2)*. For the second part, suppose that $k_1 < k_2 < k_3$ are such that $k_1, k_3 \in D_s$ for some $s \in S_t$. This implies that $\operatorname{ran} x_{k_1} \cap \operatorname{ran} f_s, \operatorname{ran} x_{k_3} \cap \operatorname{ran} f_s \neq \emptyset$, and hence $\operatorname{ran} x_{k_2} \subseteq \operatorname{ran} f_s$. This implies that $\operatorname{ran} x_{k_1} \cap \operatorname{ran} f_{s'} = \emptyset$ for every $s' \in S_t \setminus \{s\}$. Since t is a splitter, $k_2 \notin E_t$, and, by *(a.3)*, $k_2 \notin D_{s'}$ for every $s' \in S_t \setminus \{s\}$.

Let $S_t = \{s_1 < \cdots < s_d\}$ be ordered such that $f_{s_i} < f_{s_j}$ whenever $i < j$. For $k \in E_t$, the set $H_k = \{i \in [1, d] : \operatorname{ran} x_k \cap \operatorname{ran} f_{s_i} \neq \emptyset\}$ has at least two elements. We claim that the mapping $k \mapsto \max H_k \in \{2, \dots, d\}$ is one-to-one. To see this note that for $k < k'$ we obtain that $H_k \cap H_{k'} = \{\max H_k\}$ if $\max H_k = \max H_{k'}$, and $H_k < H_{k'}$ otherwise.

(b.2): Fix $s' \in S_t \setminus \{s\}$, and suppose that $\operatorname{ran} f_{s'} \cap \operatorname{ran} x_k \neq \emptyset$. Since $\operatorname{ran} f_s \cap \operatorname{ran} x_k \neq \emptyset$, we get that $\operatorname{supp} f_{s'} \cap \operatorname{ran} x_k \neq \emptyset$. By the maximality of $\mathcal{A}_k$, there is $t' \in \mathcal{A}_k$ comparable with s'. Since $\mathcal{A}_k$ is an antichain, we get that $t' \succeq s'$, and hence $t \wedge t' = u$. But f_u is of type I, a contradiction.

(b.3): This follows using *(a.4)*, and *(a.1)*, *(a.2)*. $\square$

8.1.2. *Assignments, filtrations, and their relationships.*

Definition 8.5. Given a block sequence $(x_k)_{k=1}^n$, and a regular array $\boldsymbol{\mathcal{A}} = (\mathcal{A}_k)_{k=1}^n$ for $(x_k)_{k=1}^n$, a sequence $(g_{k,t}^{\boldsymbol{\mathcal{A}}})_{t \in \mathcal{A}_k, k} \subseteq c_{00}(\mathbb{N})$ is called a *$\boldsymbol{\mathcal{A}}$-assignment* provided that $\operatorname{supp} g_{k,t} \subseteq \{k\}$ for every k and $t \in \mathcal{A}_k$. The property *(b.1)* ensures that every $\boldsymbol{\mathcal{A}}$-assignment $(g_{k,t}^{\boldsymbol{\mathcal{A}}})_{t \in \mathcal{A}_k, k}$ naturally *filters down* to the whole tree $(G_{k,t}^{\boldsymbol{\mathcal{A}}})_{t \in \mathcal{T}}$ as follows: If $k \notin D_t^{\boldsymbol{\mathcal{A}}}$, then $G_{k,t}^{\boldsymbol{\mathcal{A}}} = 0$, and if $t \in \mathcal{A}_k$, then $G_{k,t}^{\boldsymbol{\mathcal{A}}} = g_{k,t}^{\boldsymbol{\mathcal{A}}}$. Suppose that $k \in D_t^{\boldsymbol{\mathcal{A}}} \setminus D_s^{\boldsymbol{\mathcal{A}}}$. If f_t is of type I, then we define recursively $G_{k,t}^{\boldsymbol{\mathcal{A}}} = (1/w(f_t)) G_{k,s}^{\boldsymbol{\mathcal{A}}}$, where $s \in S_t$ is the unique $s = s(k,t) \in S_t$ such that $k \in D_s^{\boldsymbol{\mathcal{A}}}$ (by *(b.1)*). If f_t is of type II, $f_t = \sum_{s \in S_t} \lambda_s f_s$, then we simply set $G_{k,t}^{\boldsymbol{\mathcal{A}}} = \sum_{s \in S_t} \lambda_s G_{k,s}^{\boldsymbol{\mathcal{A}}}$. For $t \in \mathcal{T}$, let

$$G_t^{\boldsymbol{\mathcal{A}}} = \sum_{k \in D_t^{\boldsymbol{\mathcal{A}}}} G_{k,t}^{\boldsymbol{\mathcal{A}}}.$$

We call $(G_t^{\boldsymbol{\mathcal{A}}})_{t \in \mathcal{T}}$ the *filtration* of $(g_{k,t}^{\boldsymbol{\mathcal{A}}})_{t \in \mathcal{A}_k, k}$. Whenever there is no possible confusion, we write $g_{k,t}$, $G_{k,t}$ and G_t instead of the respective $g_{k,t}^{\boldsymbol{\mathcal{A}}}$, $G_{k,t}^{\boldsymbol{\mathcal{A}}}$ and $G_t^{\boldsymbol{\mathcal{A}}}$.

Proposition 8.6. *Fix $t \in \mathcal{T}$. Then we have the following:*
(c.1) For every k, $\operatorname{supp} g_{k,t} \subseteq \{k\}$. Hence $\operatorname{supp} g_t \subseteq D_t$.
(c.2) If f_t is not of type II, then $G_t = \sum_{k \in E_t} g_{k,t} + (1/w(f_t)) \sum_{s \in S_t} G_s$.
(c.3) If f_t is of type II, $f_t = \sum_{s \in S_t} \lambda_s f_s$, then $G_t = \sum_{s \in S_t} \lambda_s G_s$.

Proof. (c.1) is clear. (c.2): If f_t is of type 0, this is clear. Suppose that f_t is of type I. Then by definition

$$\begin{aligned} G_t = \sum_{k \in E_t} G_{k,t} + \sum_{k \in D_t \setminus E_t} G_{k,t} &= \sum_{k \in E_t} g_{k,t} + \sum_{s \in S_t} \sum_{k \in D_s} G_{k,t} = \\ = \sum_{k \in E_t} g_{k,t} + \sum_{s \in S_t} \frac{1}{w(f_t)} \sum_{k \in D_s} G_{k,s} &= \sum_{k \in E_t} g_{k,t} + \frac{1}{w(f_t)} \sum_{s \in S_t} G_s. \end{aligned} \tag{59}$$

(c.3): Suppose that f_t is of type II, i.e., $f_t = \sum_{s \in S_t} \lambda_s f_s$, and suppose that $k \in D_t$. Then, by (c.1), $G_t(e_k) = G_{t,k}(e_k) = \sum_{s \in S_t} \lambda_s G_{k,s}(e_k) = (\sum_{s \in S_t} \lambda_s G_s)(e_k)$. If $k \notin D_t$, then $G_t(e_k) = 0$, and $\sum_{s \in S_t} \lambda_s G_s(e_k) = 0$. $\square$

Definition 8.7. *(Canonical Assignment)* Suppose that $\boldsymbol{\mathcal{A}} = (\mathcal{A}_k)_k$ is a regular array for $(x_k)_{k=1}^n$ and $(f_t)_{t \in \mathcal{T}}$. Let $f_{k,t} = f_t(x_k) e_k^*$ for $k \in [1, n]$ and $t \in \mathcal{A}_k$. This is the *$\boldsymbol{\mathcal{A}}$-canonical assignment.*

Remark 8.8. Note that if the array $\boldsymbol{\mathcal{A}}$ is maximal, then filtering down the canonical assignment we get $f_t(w_k) = F_{k,t}(e_k)$, for every $t \in \mathcal{T}$, and $k \in D_t$: If $k \in E_t$, this is just by definition. Suppose $k \notin E_t$. If f_t is of type I, then $F_{k,t}(e_k) = (1/w(f_t)) F_{k,s}(e_k)$, where $s \in S_t$ is unique such that $k \in D_s$. By the maximality of $\mathcal{A}_k$, we get that $\operatorname{supp} f_{s'} \cap \operatorname{ran} w_k = \emptyset$ for every

$s' \in S_t \setminus \{s\}$ (by *(b.2)*), hence $f_t(x_k) = (1/w(f_t))f_s(x_k) = (1/w(f_t))F_{k,s}(e_k) = F_{k,t}(e_k)$, by the inductive hypothesis. If $f_t = \sum_{s\in S_t} \lambda_s f_s$ is of type II, then by the maximality of $\mathcal{A}_k$, $f_t(x_k) = \sum_{s\in S_t,k\in D_s} \lambda_s f_s(x_k) = \sum_{s\in S_t,k\in D_s} \lambda_s F_{k,t}(e_k) = F_{k,t}(e_k)$, the last equality because $F_{k,u} = 0$ if $k \notin D_u$.

We obtain that $f_t(\sum_{k\in D_t} b_k x_k) = F_t(\sum_{k\in D_t} b_k e_k) = F_t(\sum_{k=1}^n b_k e_k)$ for every sequence of scalars $(b_k)_{k=1}^n$. The last equality follows from $\operatorname{supp} G_t \subseteq D_t$. In particular, $f(\sum_{k=1}^n b_k x_k) = G_\emptyset(\sum_{k=1}^n b_k e_k)$, since $D_\emptyset = \{k : \operatorname{supp} f \cap \operatorname{ran} x_k \neq \emptyset\}$, by maximality of $\boldsymbol{\mathcal{A}}$.

Proposition 8.9. *Suppose that $\boldsymbol{\mathcal{A}} = (\mathcal{A}_k)_{k=1}^n$ is a regular array (not necessarily maximal) for $(x_k)_{k=1}^n$ and $(f_t)_{t\in\mathcal{T}}$. Fix scalars $(b_k)_{k=1}^n$, $(c_k)_{k=1}^n$ and suppose that $(g_{k,t})_{t\in\mathcal{A}_k,k}$, $(h_{k,t})_{t\in\mathcal{A}_k,k}$ are $\boldsymbol{\mathcal{A}}$-assignments.*
(1) If for every $t \in \mathcal{A}_k$ $g_{k,t}(b_k e_k) \le h_{k,t}(c_k e_k)$, then for every $t \in \mathcal{T}$, $G_{k,t}(b_k e_k) \le H_{k,t}(c_k e_k)$.
(2) $\|G_{k,u}(e_k)\|_\infty \le \max\{\|g_{k,t}\|_\infty : t \in \mathcal{A}_k\}$, for every $u \in \mathcal{T}$.
(3) If for every $t \in \bigcup_{k=1}^n \mathcal{A}_k$ $\sum_{k\in E_t} g_{k,t}(b_k e_k) \le \sum_{k\in E_t} h_{k,t}(c_k e_k)$, then for every $t \in \mathcal{T}$, $G_t(\sum_{k\in D_t} b_k e_k) \le H_t(\sum_{k\in D_t} c_k e_k)$.
(4) $\|G_u\|_\infty \le \|\sum_{t\in A_k,k} g_{k,t}\|_\infty$ for every $u \in \mathcal{T}$.

PROOF. This follows from Proposition 8.6. □

8.1.3. *Two successive filtrations.* In some applications of the theory one needs to do the process of assignment and filtration twice starting with different arrays of antichains. To see this, suppose that $\boldsymbol{\mathcal{C}}$ and $\boldsymbol{\mathcal{D}}$ are regular arrays for $(x_k)_{k=1}^n$ and $(f_t)_{t\in\mathcal{T}}$. Then we can naturally define a $\boldsymbol{\mathcal{D}}$-assignment $(g_{k,t}^{\boldsymbol{\mathcal{D}}})_{t\in\mathcal{D}_k,k}$ by taking the filtration $g_{k,t}^{\boldsymbol{\mathcal{D}}} = G_{k,t}^{\boldsymbol{\mathcal{C}}}$. For this to work, one needs the following special relationship between $\boldsymbol{\mathcal{C}}$ and $\boldsymbol{\mathcal{D}}$.

Definition 8.10. We write $\boldsymbol{\mathcal{C}} \not\prec \boldsymbol{\mathcal{D}}$ if for every k, every $c \in \mathcal{C}_k$ and every $d \in \mathcal{D}_k$, we have that $c \not\prec d$. A $\boldsymbol{\mathcal{C}}$-assignment $(g_{k,t}^{\boldsymbol{\mathcal{C}}})_{k\in\mathcal{C}_k,k}$ is called *coherent* provided that $g_{k,t}^{\boldsymbol{\mathcal{C}}} = 0$ whenever $f_t(w_k) = 0$.

Proposition 8.11. *Suppose that $\boldsymbol{\mathcal{C}} \not\prec \boldsymbol{\mathcal{D}}$ are two regular arrays for $(x_k)_{k=1}^n$ and $(f_t)_{t\in\mathcal{T}}$, and suppose that $\boldsymbol{\mathcal{D}}$ is in addition maximal. Fix a coherent $\boldsymbol{\mathcal{C}}$-assignment $(g_{k,t}^{\boldsymbol{\mathcal{C}}})_{t\in\mathcal{C}_k,k}$. Then,*
(a) For every $k \in D_t^{\boldsymbol{\mathcal{C}}} \cap D_t^{\boldsymbol{\mathcal{D}}}$, $G_{k,t}^{\boldsymbol{\mathcal{C}}} = G_{k,t}^{\boldsymbol{\mathcal{D}}}$.
(b) $g_\emptyset^{\boldsymbol{\mathcal{C}}} = g_\emptyset^{\boldsymbol{\mathcal{D}}}$.

PROOF. (a): If $k \in E_t^{\boldsymbol{\mathcal{D}}}$, this is just by definition. Suppose f_t is of type I, and suppose that $k \in D_s^{\boldsymbol{\mathcal{D}}}$, for some $s \in S_t$. Then $G_{k,t}^{\boldsymbol{\mathcal{D}}} = (1/w(f_t))G_{k,s}^{\boldsymbol{\mathcal{D}}}$. Since $\boldsymbol{\mathcal{D}}$ is a maximal regular array, by Proposition 8.4 *(b.2)*, $\operatorname{ran} f_{s'} \cap \operatorname{ran} w_k = \emptyset$ for every $s' \in S_t \setminus \{s\}$. If $k \in D_s^{\boldsymbol{\mathcal{C}}}$, then we are done by the inductive hypothesis. So, suppose $k \in E_t^{\boldsymbol{\mathcal{C}}}$, i.e., $t \in \mathcal{C}_k$. Hence, $t \precneqq u$ for some $u \in \mathcal{D}_k$ (because $k \in D_s^{\boldsymbol{\mathcal{D}}}$), contradicting our assumption that $\boldsymbol{\mathcal{C}} \not\prec \boldsymbol{\mathcal{D}}$. If $f_t = \sum_{s\in S_t} \lambda_s f_s$ is a sub-convex combination, then

$$G_{k,t}^{\boldsymbol{\mathcal{D}}} = \sum_{s\in S_t,k\in D_s^{\boldsymbol{\mathcal{D}}}} G_{k,s}^{\boldsymbol{\mathcal{D}}} = \sum_{s\in S_t,k\in D_s^{\boldsymbol{\mathcal{D}}}\cap D_s^{\boldsymbol{\mathcal{C}}}} G_{k,s}^{\boldsymbol{\mathcal{D}}} = \sum_{s\in S_t,k\in D_s^{\boldsymbol{\mathcal{D}}}\cap D_s^{\boldsymbol{\mathcal{C}}}} G_{k,s}^{\boldsymbol{\mathcal{D}}} = G_{k,t}^{\boldsymbol{\mathcal{C}}}. \tag{60}$$

To see the last equality note that if $k \notin D_s^{\boldsymbol{\mathcal{D}}}$, then, by the maximality of $\boldsymbol{\mathcal{D}}$, $\operatorname{supp} f_u \cap \operatorname{ran} w_k = \emptyset$ for every $u \succeq s$, so, by the coherence of the assignment, $G_{k,t}^{\boldsymbol{\mathcal{C}}} = 0$; if $k \notin D_s^{\boldsymbol{\mathcal{C}}}$, then $k \notin D_u^{\boldsymbol{\mathcal{C}}}$ for all $u \succeq s$, and so $g_{k,u}^{\boldsymbol{\mathcal{C}}} = 0$ for all $u \succeq s$ $u \in \mathcal{C}_k$, giving us $G_{k,s}^{\boldsymbol{\mathcal{D}}} = 0$.

(b): Note now that

$$g_\emptyset^{\boldsymbol{\mathcal{D}}} = \sum_{k\in D_\emptyset^{\boldsymbol{\mathcal{C}}}} g_{k,\emptyset}^{\boldsymbol{\mathcal{D}}} = \sum_{k\in D_\emptyset^{\boldsymbol{\mathcal{C}}}\cap D_\emptyset^{\boldsymbol{\mathcal{C}}}} g_{k,\emptyset}^{\boldsymbol{\mathcal{D}}} = g_\emptyset^{\boldsymbol{\mathcal{C}}}. \tag{61}$$

For if $k\in D_\emptyset^{\boldsymbol{\mathcal{C}}}\setminus D_\emptyset^{\boldsymbol{\mathcal{D}}}$, then by the maximality of $\boldsymbol{\mathcal{D}}$, for all $u\in\mathcal{T}$, $\operatorname{supp} f_u\cap\operatorname{ran} w_k=\emptyset$, hence, by the coherence of the $\boldsymbol{\mathcal{C}}$-assignment $g_{k,u}^{\boldsymbol{\mathcal{C}}}=0$ for all u, and hence $g_{k,\emptyset}^{\boldsymbol{\mathcal{D}}}=0$; if $k\in D_\emptyset^{\boldsymbol{\mathcal{D}}}\setminus D_\emptyset^{\boldsymbol{\mathcal{C}}}$, then every $\mathcal{C}_k=\emptyset$, and so $g_{k,\emptyset}^{\boldsymbol{\mathcal{C}}}=0$. □

Let us now give the two main applications of this general theory of tree-analysis.

8.2. **The proof of the basic inequality.** Recall that W is the minimal subset of $c_{00}(\mathbb{N})$ containing $\{\pm e_k^*\}_k$, and closed under (m_j^{-1}, n_{4j})-operations. Fix a (C,ε)-RIS $(x_k)_{k=1}^n$, and fix $(j_k)_{k=1}^n$ witnessing that $(x_k)_{k=1}^n$ is a (C,ε)-RIS, i.e.,
a) $\|x_k\|\le C$,
b) $|\operatorname{supp} x_k|\le m_{j_{k+1}}\varepsilon$ and
c) For all type I functionals ϕ of K with $w(\phi)<m_{j_k}$, $|\phi(x_k)|\le C/w(\phi)$. Fix a sequence $(b_k)_{k=1}^n$ of scalars, $\max_k|b_k|\le 1$, and $f\in K_{\omega_1}$. Let $(f_t)_{t\in\mathcal{T}}$ be a tree-analysis of f. Consider the maximal regular array $\boldsymbol{\mathcal{A}}=(\mathcal{A}(x_k,\mathrm{C}))_{k=1}^n$, where C is the set of nodes t such that f_t is of type I and $w(f_t)=m_{j_0}$.

We introduce the following two $\boldsymbol{\mathcal{A}}$-assignments $(g_{k,t})_{t\in\mathcal{A}_k,k}$, and $(r_{k,t})_{t\in\mathcal{A}_k,k}$. Fix k and $t\in\mathcal{A}_k$. If t_t is of type 0, then we set $g_{k,t}=e_k^*$ and $r_{k,t}=0$. Suppose that t is of type I, and $w(f_t)\neq m_{j_0}$. Let

$$l_t=\min\{k\in E_t\,:\,w(f_t)\le m_{j_k}\} \tag{62}$$

if this exists, and $l_t=\infty$ otherwise. Then let

$$g_{k,t}=\begin{cases}\frac{1}{w(f_t)}e_k^* & \text{if } k>l_t\\ 0 & \text{if } k<l_t\\ e_k^* & \text{if } k=l_t\end{cases}\quad r_{k,t}=\begin{cases}0 & \text{if } k>l_t\\ \varepsilon e_k^* & \text{if } k<l_t\\ 0 & \text{if } k=l_t\end{cases}$$

Suppose now that $w(f_t)=m_{j_0}$. Notice that E_t is an interval. Set

$$k_t=\max\{l\in D_t\,:\,|b_l|=\|(b_i)_{i\in E_t}\|_\infty\}. \tag{63}$$

Then let

$$g_{k,t}=\begin{cases}e_k^* & \text{if } k=k_t\\ 0 & \text{if } k\neq k_t\end{cases}\quad r_{k,t}=\varepsilon e_k^*$$

Let $(G_t)_{t\in\mathcal{T}}$ and $(R_t)_{t\in\mathcal{T}}$ be the corresponding filtrations.

Claim (D). *Fix $t\in\mathcal{T}$. Then:*
(d.1) $\|R_t\|_\infty\le\varepsilon$.
(d.2) $|f_t(\sum_{k\in D_t}b_kx_k)|\le C(G_t+R_t)(\sum_{k\in D_t}|b_k|e_k)$.
(d.3) For every t for which f_t is of type I, either $G_t\in\operatorname{conv}\{h\in W\,:\,w(h)=w(f_t)\}$ or $G_t=e_k^*+h_t$ for some $k\notin\operatorname{supp} h_t$ and $h_t\in\operatorname{conv}\{h\in W\,:\,w(h)=w(f_t)\}$.

Proof of Claim: (d.1) follows from Proposition 8.9, and *(d.2)* follows also from the same proposition applied to the canonical $\boldsymbol{\mathcal{A}}$-assignment, the assignment $(C(g_{k,t}+h_{k,t}))_{t\in\mathcal{A}_k,k}$, and the sequences of scalars $(b_k)_k$ and $(|b_k|)_k$.

(d.3): If $w(f_t)=m_{j_0}$, then t is a catcher and $G_t=\sum_{k\in E_t}g_{k,t}=e^*_{k_t}\in W$. Suppose that t is of type I, $w(f_t)\neq m_{j_0}$. By *(c.2)* and the particular $\boldsymbol{\mathcal{A}}$-assignment, we know that either $G_t=(1/w(f_t))(\sum_{k\in E_t,k>l_t}e^*_k+\sum_{s\in S_t}G_s)$ or $G_t=e^*_{l_t}+h_t$, where $h_t=(1/w(f_t))(\sum_{k\in E_t,k>l_t}e^*_k+\sum_{s\in S_t}G_s)$. Assume this last case holds.
Subcase 1a. For every $s\in S_t$ the functional f_s is not of type II. From the inductive hypothesis, we have that for every $s\in S_t$, $G_s=h_s$ or $G_s=e^*_{l_s}+h_s$, $h_s\in W$. For $s\in S_t$ such that $G_s=e^*_{l_s}+h_s$, set $I^1_s=\{n\in\mathbb{N}:n<l_s\}$ and $I^2_s=\{n\in\mathbb{N}:n>l_s\}$. We set $h^1_s=I^1_sh_s$, $h^2_s=I^2_sh_s$. Then, for every $s\in S_t$ the functionals h^1_s, $e^*_{l_s}$, and h^2_s are successive and belong to W. By *(b.1)*, for $s\neq s'\in S_t$ the corresponding functionals together with $\{e^*_k\}_{k\in E_t,k>l_t}$ form a block family, and we obtain that

$$\#\{e^*_k\}_{k\in E_t,k>l_t}+\#\{e^*_{l_s}\,:\,s\in S_t\}+\#\{h^1_s\,:\,s\in S_t\}+\#\{h^2_s\,:\,s\in S_t\}\leq 4\#S_t. \tag{64}$$

Therefore, $(1/w(f_t))(\sum_{k\in E_t,k>l_t}e^*_k+\sum_{s\in S_t}G_s)\in W$.
Subcase 1b. There are $s\in S_t$ for which f_s is of type II. Let B_1 be the set of immediate successors s of t such that f_s is of type II, and $B_2=S_t\setminus S_1$. Observe that every sub-convex combination $f_s=\sum_{u\in S_s}r_uf_u$ satisfies that f_u is of type I. We may assume, allowing repetitions if needed, that for every $s\in S_t$ such that f_s is of type II, $f_s=(1/k)\sum_{q=1}^k f_{s,q}$, where each $f_{s,q}\in\{f_u\,:\,u\in S_s\}$. For each $q=1,2,\dots,k$ we set $h^q_t=(1/m_j)(\sum_{l\in E_t,l>l_t}e^*_l+\sum_{s\in B_1}G_s+\sum_{s\in B_2}G_{s,q})$, where $G_{s,q}=G_u$ for $u\in S_s$ such that $f_{s,q}=f_u$. A similar argument as in the previous subcase shows that $h^q_t\in W$ with $w(h^q_t)=m_j$ for $q=1,2,\dots,k$ and $h_t=(1/k)\sum_{q=1}^k h^q_t$, as required. □

The particular case $t=\emptyset$, the root of $\mathcal{T}$, gives us the conclusion of the Basic Inequality.

REMARK 8.12. Note that a finer assignment using the same array of antichains will actually give us the conclusion of the Basic Inequality for a bit smaller auxiliary space $T[(m_j^{-1},2n_j)_j]$.

8.3. **The proof of the finite interval representability of J_{T_0}.** The general scheme of the proof is quite similar to the proof of Basic Inequality though the input block sequence of vectors is slightly differently chosen. Notice however that the finite interval representability involves two inequalities needed for showing that the representing operator as well as its inverse are uniformly bounded. Thus, while in the proof of the Basic Inequality we could afford to go the auxiliary space $T[(m_j^{-1},4n_j)_j]$ this is no longer possible in this case. In other words, we need to improve on the counting inequality (64). It is exactly for this reason that we introduce below two arrays of antichains and use two successive filtrations as explained above in Subsection 8.1.3.

Fix a transfinite block sequence $(x_\alpha)_{\alpha<\gamma}$, $n\in\mathbb{N}$, a sequence $I_1\leq I_2\leq\cdots\leq I_n$ of successive, not necessarily distinct, infinite intervals of γ, and $\varepsilon>0$. Let j_0 be such that $m_{2j_0+1}>100n/\varepsilon$ and set $l=n_{2j_0+1}/m_{2j_0+1}$. Find a $(1,j_0)$-dependent sequence $(z_1,\psi_1,\dots,z_{n_{2j_0+1}},\psi_{n_{2j_0+1}})$ such that (a) $\operatorname{ran}\psi_i\subseteq\operatorname{ran}z_i$ for every $i=1,\dots,n_{2j_0+1}$ and (b) $(z_k)_{k=(i-1)l+1}^{il}\subseteq\langle x_\alpha\rangle_{\alpha\in I_i}$ for every

$i = 1, \dots, n$. Let

$$\phi = \frac{1}{m_{2j_0+1}} \sum_{i=1}^{n_{2j_0+1}} \psi_i,$$

and for each $k = 1, \dots, n$ we set

$$w_k = \frac{m_{2j_0+1}}{l} \sum_{i=(k-1)l+1}^{kl} z_i \text{ and } \phi_k = \frac{1}{m_{2j_0+1}} \sum_{i=(k-1)l+1}^{kl} \psi_i \in K_{\omega_1}.$$

Proposition 8.13. *Fix $k = 1, \dots, n$. Then*
(1) $\operatorname{ran}\phi_k \subseteq \operatorname{ran} w_k$*,* $\phi_k w_k = 1$ *and* $1 \leq \|w_k\| \leq 24$*.*
(2) For every $f \in K_{\omega_1}$ of type I with $w(f) > m_{2j_0+1}$, $|f(w_k)| \leq 1/m_{2j_0+1}^2$.
(3) Let $f \in K_{\omega_1}$ be of type I, $f = (1/w(f)) \sum_{i=1}^{d} f_i$ with $w(f) = m_{2j+1}$ for $j < j_0$ and $d \leq n_{2j+1}$. Let $d_0 = \max\{i \leq d : w(f_i) < m_{2j_0+1}\}$, and set $f_{\mathrm{L}} = 1/m_{2j+1} \sum_{i=1}^{d_0-1} f_i$ and $f_{\mathrm{R}} = 1/m_{2j+1} \sum_{i=d_0+1}^{d} f_i$. Then $|f_{\mathrm{L}}(w_k)| \leq 1/m_{2j_0+1}^2$ and $|f_{\mathrm{R}}(w_k)| \leq 1/m_{2j_0+1}$.
(4) Let $f = (1/w(f)) \sum_{i=1}^{d} f_i$ with $w(f) = m_{2j_0+1}$ and $d \leq n_{2j_0+1}$ be such that $\#\{i \in [1, d] : w(f_i) = w(\psi_i)$ and $\operatorname{supp} z_i \cap \operatorname{supp} f_i \neq \emptyset\} \leq 2$. Then, $|f(w_k)| \leq 1/m_{2j_0+1}^2$.

Proof. First of all, note that $(z_i)_{i=(k-1)l+1}^{kl}$ is a $(12, 1/n_{2j_0+1})$-RIS. Note also that *(1)* and *(2)* follow from Proposition 4.7. *(3)* By the properties of special sequences,

$$\# \bigcup_{i=1}^{d_0-1} \operatorname{supp} f_i \leq w(f_{d_0}) < m_{2j_0+1}. \tag{65}$$

So, $|f_{\mathrm{L}}(w_k)| \leq \|f_0\|_{\ell_1} \|w_k\|_\infty \leq m_{2j_0+1}^3 / n_{2j_0+1} \leq 1/m_{2j_0+1}^2$. Let us now estimate $|f_{\mathrm{R}}(w_k)|$. To save on notation we only estimate for $k = 1$. Set

$$F_0 = \{r \in [1, l] : \#(\{i \in [d_0 + 1, d] : \operatorname{ran} z_r \cap \operatorname{supp} f_i \neq \emptyset\}) \geq 2\},\ F_1 = [1, l] \setminus F_0.$$

Notice that $|F_0| \leq n_{2j+1} - 1$. For $i = 0, 1$ let $w^i = (m_{j_1}/l) \sum_{k \in F_i} z_k$. Since $f_{\mathrm{R}} \in K_{\omega_1}$ and since $(z_k)_k$ is a $(12, 1/n_{2j_0+1})$-RIS, we have that

$$|f_{\mathrm{R}}(w^0)| \leq \|w^0\| \leq \frac{m_{2j_0+1}}{l} \sum_{k \in F_0} \|z_k\| \leq \frac{m_{2j_0+1}}{l} 6 n_{2j+1}. \tag{66}$$

To estimate $|f_{\mathrm{R}}(w^1)|$ we use the basic inequality. For each $i = d_0 + 1, \dots, d$, let

$$H_i = \{k \in F_1 : \operatorname{ran} z_k \cap \operatorname{supp} f_i \neq \emptyset\}.$$

Note that $\{H_i\}_i$ is a partition of F_1 and is a block family. For $i = d_0 + 1, \dots, d$, we set $w^{1,i} = m_{j_1}/l \sum_{k \in H_i} z_k$. Clearly $w^1 = w^{1,d_0+1} + \dots + w^{1,d}$ and hence

$$|f_{\mathrm{R}}(w^1)| \leq \sum_{i=d_0+1}^{d} |f_{\mathrm{R}}(w^{1,i})| = \frac{1}{m_{2j+1}} \sum_{i=d_0+1}^{d} |f_i(w^{1,i})|. \tag{67}$$

Let us estimate now $|f_i(w^{1,i})|$, for $i = d_0 + 1, \dots, d$. For a fixed such i, applying again the basic inequality, we obtain that $|f_i(w^{1,i})| \leq 12(g_1^i + g_2^i)(m_{2j_0+1}/l \sum_{k \in H_i} e_k)$, where in the worst case, $g_1^i = h^i + e_{k_i}^*$, with $h^i \in W$, and $h^i \in \operatorname{conv}_{\mathbb{Q}}\{h \in W : w(h) = w(f_i)\}$. Since the auxiliary space is 1-unconditional, by Proposition 4.6, $|h^i((m_{2j_0+1}/l)\sum_{k\in H_i} e_k)| \leq m_{2j_0+1}/w(f_i)$. Note that $\|g_2^i\|_\infty \leq 1/n_{2j_0+1}$. Putting all these inequalities together we get

$$\begin{aligned}|f_{\mathrm{R}}(w^1)| \leq & \frac{12}{m_{2j+1}}\left(\sum_{i=d_0+1}^{d} \frac{m_{2j_0+1}}{w(f_i)} + \frac{m_{2j_0+1}n_{2j+1}}{l} + \frac{m_{2j_0+1}}{n_{2j_0+1}}\right) \leq \\ \leq & \frac{12}{m_{2j+1}}\left(\sum_{i=d_0+1}^{d} \frac{m_{2j_0+1}}{w(f_i)} + \frac{m_{2j_0+1}^2 n_{2j+1}}{n_{2j_0+1}} + \frac{m_{2j_0+1}}{n_{2j_0+1}}\right). \end{aligned} \tag{68}$$

Using (66) and (68) we obtain

$$|f_{\mathrm{R}}(w_1)| \leq \frac{12m_{2j_0+1}}{m_{2j+1}}\left(\frac{2n_{2j+1}m_{2j_0+1}}{n_{2j_0+1}} + \frac{1}{n_{2j_0+1}} + \sum_{i=d_0+1}^{d} \frac{1}{w(f_i)}\right) \leq \frac{1}{m_{2j_0+1}}. \tag{69}$$

(4) Let $E = \{i \in [1,d] \,:\, w(f_i) = w(\psi_i)$ and $\operatorname{supp} z_i \cap \operatorname{supp} f_i \neq \emptyset\}$. By our assumptions, $\#E \leq 2$. For $i \in [(k-1)l, kl] \setminus E$ the properties of the dependent sequences yield that $|f(z_i)| \leq 1/n_{2j_0+1}$. Hence, $|f(w_k)| \leq 2 \cdot 24m_{2j_0+1}/l + m_{2j_0+1}/n_{2j_0+1} \leq 1/m_{2j_0+1}^2$. □

Lemma 8.14. *For the above defined sequence $(w_k)_k$ we have that*

$$\|\sum_{k=1}^{n} b_k w_k\| \leq 121\|\sum_{k=1}^{n} b_k v_k\|_{J_{T_0}} \tag{70}$$

for every choice of scalars $(b_k)_{k=1}^n$.

Proof. Fix a sequence $(b_k)_{k=1}^n$ of scalars with $\max_k |b_k| \leq 1$, an $f \in K_{\omega_1}$, and its tree $(f_t)_{t\in\mathcal{T}}$.
Antichains. A node $t \in \mathcal{T}$ is called *relevant* if *(1)* $w(f_t) \leq m_{2j_0+1}$, and *(2)* if $u \precneqq t$ is its immediate predecessor, if f_u is of type I, and if $w(f_u) = m_{2j+1} < m_{2j_0+1}$, then $t = s(u) = \max\{s \in S_u \,:\, w(f_s) < m_{2j_0+1}\}$, where the maximum is taken according to the block ordering $S_u = \{s_1 < \cdots < s_d\}$. Let C be the set of nodes t which are either non-relevant, or such that f_t is of type I and $w(f_t) = m_{2j_0+1}$. Let $\boldsymbol{\mathcal{B}} = (\mathcal{B}_k)_{k=1}^n$ where $\mathcal{B}_k = \mathcal{A}(w_k, \mathrm{C})$ for $k = 1, \ldots, n$ (see *(A.1)-(A.5)* in Proposition 8.3 above). For each k, let $\mathcal{B}_k^{\mathrm{unc}} = \mathrm{S}(\mathcal{B}_k) \setminus \mathrm{C}$ be the set of splitters that are not in C, $\mathcal{B}_k^{\mathrm{cnd}} = \mathcal{B}_k \cap \mathrm{C}$, and $\mathcal{B}_k^{\mathrm{at}} = \mathcal{B}_k \setminus (\mathcal{B}_k^{\mathrm{unc}} \cup \mathcal{B}_k^{\mathrm{cnd}})$.

Fix $u \in \mathcal{B}_k^{\mathrm{unc}}$, and observe that u is a splitter of x_k for every $k \in E_u$. List all $s \in S_u$ such that $\operatorname{ran} f_s \cap \operatorname{ran} w_k \neq \emptyset$,$\{s_{k,1}, \ldots, s_{k,d}\}$ ordered according to the block ordering $f_{s_{k,1}} < \cdots < f_{s_{k,d}}$. Set

$$\begin{aligned} w_{k,u}^{\mathrm{in}} =& w_k|[\min\operatorname{supp} w_k, \max\operatorname{supp} f_{s_{k,1}}] \\ w_{k,u}^{\mathrm{fin}} =& w_k - w_{k,t}^{\mathrm{in}}. \end{aligned}$$

For $\star \in \{\mathrm{in}, \mathrm{fin}\}$, let $\mathcal{B}_{k,u}^{\star} = \mathcal{A}(w_{k,u}^{\star}, \mathrm{C}^{nr})$, where C^{nr} is the set of non-relevant nodes of $\mathcal{T}$. Set $\mathcal{B}_k^{\star} = \bigcup_{u\in\mathcal{B}_k^{\mathrm{unc}}} \mathcal{B}_{k,u}^{\star}$. Observe that $\boldsymbol{\mathcal{B}}^{\star} = (\mathcal{B}_k^{\star})_{k=1}^n$ is a regular (not necessarily maximal) arrow for $(w_k)_{k=1}^n$ and $(f_t)_{t\in\mathcal{T}}$, whenever $\star \in \{\mathrm{in}, \mathrm{fin}, \mathrm{cnd}, \mathrm{at}\}$.
Assignments and filtrations. Consider the following $\boldsymbol{\mathcal{B}}^{\star}$-assignments $(g_{k,t}^{\star})_{k\in\mathcal{B}_k^{\star},k}$ where $\star \in \{\mathrm{in}, \mathrm{fin}, \mathrm{cnd}\}$, and $(r_{k,t}^{\star})_{k\in\mathcal{B}_k^{\star},k}$ where $\star \in \{\mathrm{in}, \mathrm{fin}, \mathrm{cnd}, \mathrm{at}\}$: Fix k, and $t \in \bigcup_{\star\in\{\mathrm{in},\mathrm{fin},\mathrm{cnd},\mathrm{at}\}} \mathcal{B}_k^{\star}$.
(a) Suppose that f_t is of type 0. Then we set $r_{k,t}^{\mathrm{at}} = (1/m_{2j_0+1})e_k^*$ if $t \in \mathcal{B}_k^{\mathrm{at}}$, and we set $g_{k,t}^{\star} = 0$ and $r_{k,t}^{\star} = (1/m_{2j_0+1})e_k^*$, if $t \in \mathcal{B}_k^{\star}$ for some $\star \in \{\mathrm{in}, \mathrm{fin}\}$.

(b) Suppose that t is non-relevant. Then clearly $t\notin \mathcal{B}_k^{\mathrm{at}}$. Fix $\star\in\{\mathrm{in},\mathrm{fin},\mathrm{cnd}\}$. We set $g_{k,t}^\star=0$ in all cases. Suppose that $w(f_t)>m_{2j_0+1}$. Then we set $r_{k,t}^\star=(1/m_{2j_0+1})e_k^*$ for $\star\in\{\mathrm{in},\mathrm{fin}\}$, and $r_{k,t}^{\mathrm{cnd}}=(\mathrm{sgn}(b_k)/m_{2j_0+1})e_k^*$. Finally, if $t\neq s(u)$, where u is the immediate predecessor of t (see the definition of relevant node), then we set $r_{k,t}^\star=\|f_t(w_k)\|e_k^*$ for $\star\in\{\mathrm{in},\mathrm{fin}\}$ and $r_{k,t}^{\mathrm{cnd}}=\mathrm{sgn}(b_k)\|f_t(w_k)\|e_k^*$.
(c) Suppose now that t is relevant. There are two subcases.
(c.1) $w(f_t)=m_{2j_0+1}$. If $t\in\mathcal{B}_k^\star$, for $\star\in\{\mathrm{in},\mathrm{fin}\}$, then we set $g_{k,t}^\star=(1/w(f_t))e_k^*$ and $r_{k,t}^\star=0$. Suppose that $t\in\mathcal{B}_k^{\mathrm{cnd}}$. Suppose that $f_t=\pm I(1/m_{2j_0+1})\sum_{i=1}^{n_{2j_0+1}}g_i$, where $I\subseteq\omega_1$ is an interval, and $\Phi=(g_1,\dots,g_{n_{2j_0+1}})$ is a $2j_0+1$-special sequence. Set $\Psi=(\psi_1,\dots,\psi_{n_{2j_0+1}})$. Consider $I_t=\{i\in[1,\kappa_{\Phi,\Psi}-1]\,:\,Ig_i\neq 0\}=[k(t,1),k(t,2)]$, and let $\varepsilon_t=\mathrm{sgn}(\sum_{k=k(t,1)+1}^{k(t,2)-1}b_k)$. If $k=k(t,i)$ for $i=1,2$, then we set $g_{k,t}^{\mathrm{cnd}}=\mathrm{sgn}\,(b_{k(t,i)})e_{k(t,i)}^*$ and $r_{k,t}^{\mathrm{cnd}}=0$. We set $g_{k,t}^{\mathrm{cnd},\mathcal{B}_k^{\mathrm{cnd}}}=\varepsilon_te_k^*$ and $r_{k,t}^{\mathrm{cnd},\mathcal{B}_k^{\mathrm{cnd}}}=0$ if $k\in(k(t,1),k(t,2))$. We set $g_{k,t}^{\mathrm{cnd}}=0$, and $r_{k,t}^{\mathrm{cnd}}=\mathrm{sgn}(b_k)(1/m_{2j_0+1})e_k^*$ otherwise.
(c.2) Suppose that $w(f_t)\neq m_{2j_0+1}$. Then $t\in\mathcal{B}_k^\star$, for some $\star\in\{\mathrm{in},\mathrm{fin}\}$. Set $g_{k,t}^\star=(1/w(f_t))e_k^*$ and $r_{k,t}^\star=0$ for all cases, except for $w(f_t)=m_{2j+1}<m_{2j_0+1}$. In this case, we observe that since t is splitting there are at least two immediate successor $s_1\neq s_2\in S_t$ such that $\mathrm{ran}\,w_{k,u}^\star\cap\mathrm{ran}\,f_{s_i}\neq\emptyset$ $(i=1,2)$ for some $u\in\mathcal{B}_k^\star$. This implies that there is at most one $k\in E_t^{\mathcal{B}^\star}$ such that $\mathrm{ran}\,f_{s(t)}\cap\mathrm{ran}\,w_{k,v}^\star\neq\emptyset$ for $v\in\mathcal{B}_k^{\mathrm{unc}}$, and $t\in\mathcal{B}_{k,v}^\star$. Then we set $g_{k,t}^\star=(1/m_{2j_0+1})e_k^*$ and $r_{k,t}^\star=0$ if k is this one, and $g_{k,t}^\star=0$ and $r_{k,t}^\star=(1/m_{2j_0+1})e_k^*$ otherwise.

Let $(G_t^\star)_{t\in\mathcal{T}}$, $(R_t^\star)_{t\in\mathcal{T}}$ be the corresponding filtrations. Recall that given a regular array $\boldsymbol{\mathcal{A}}=(\mathcal{A}_k)_k$ for $(x_k)_k$ and $(f_t)_{t\in\mathcal{T}}$ the canonical $\boldsymbol{\mathcal{A}}$-assignment $(f_{k,t}^{\boldsymbol{\mathcal{A}}})_{t\in\mathcal{A}_k,k}$ is defined by $f_{k,t}^{\boldsymbol{\mathcal{A}}}=f(x_k)e_k^*$. It was shown in Remark 8.7 that if in addition $\boldsymbol{\mathcal{A}}$ is maximal, then for every $(a_k)_{k=1}^n$ and every $t\in\mathcal{T}$, $F^{\boldsymbol{\mathcal{A}}}(\sum_{k\in D_t^{\boldsymbol{\mathcal{A}}}}a_ke_k)=f_t(\sum_{k\in D_t^{\boldsymbol{\mathcal{A}}}}a_kw_k)$.

Claim. *Fix $t\in\mathcal{T}$, and for $\star\in\{\mathrm{in},\mathrm{fin},\mathrm{cnd},\mathrm{at}\}$ let $D_t^\star=D_t^{\boldsymbol{\mathcal{B}}^\star}$. Then:*
(e.1) $|F_t^\star(\sum_{k\in D_t^\star}b_ke_k)|\le 24(G_t^\star+R_t^\star)(\sum_{k\in D_t^\star}|b_k|e_k)$ *for* $\star\in\{\mathrm{in},\mathrm{fin}\}$.
(e.2) $|F_t^{\mathrm{cnd}}(\sum_{k\in D_t^{\mathrm{cnd}}}b_ke_k)|\le 24(G_t^{\mathrm{cnd}}+R_t^{\mathrm{cnd}})(\sum_{k\in D_t^{\mathrm{cnd}}}b_ke_k)$.
(e.3) $|F_t^{\mathrm{at}}(\sum_{k\in D_t^{\mathrm{at}}})b_ke_k|\le 24|R_t^{\mathrm{at}}(\sum_{k\in D_t^{\mathrm{at}}})b_ke_k|$.
(e.4) $G_t^\star\in W(T_0)$ *for* $\star\in\{\mathrm{in},\mathrm{fin}\}$*, and* $G_t^{\mathrm{cnd}}\in 3W(J_{T_0})$.
(e.5) $\|R_t^{\mathrm{at}}\|_\infty\le 1/m_{2j_0+1}$. *For* $\star\in\{\mathrm{in},\mathrm{fin},\mathrm{cnd}\}$*, either t is non-relevant,* $w(f_t)<m_{2j_0+1}$ *and* $G_k^\star=\sum_{k\in E_t^{\mathcal{B}^\star}}\|f_t(w_k)\|e_k^*$ *or* $\|R_t^\star\|_\infty\le 1/m_{2j_0+1}$.

Proof of Claim: (e.1)-(e.3) are immediate applications of Proposition 8.9. *(e.4)*: Most of the cases follow immediately by definition of the corresponding assignments. We sketch the non-trivial ones: Suppose that t is relevant. If $w(f_t)=m_{2j_0+1}$, then $D_t^{\mathrm{cnd}}=E_t^{\mathrm{cnd}}$, and the corresponding assignment gives that $G_t^{\mathrm{cnd}}=\lambda_1e_{k(t,1)}^*+\lambda_2e_{k(t,2)}^*+\varepsilon_t\sum_{k\in E_t^{\mathrm{cnd}}\cap(k(t,1),k(t,2))}e_k^*\in 3W(J_{T_0})$, where $\lambda_i=\mathrm{sgn}(b_{k(t,i)})\chi_{E_t^{\mathrm{cnd}}}(k(t,i))$, for $i=1,2$, and where χ_E denotes the characteristic function of E. Fix $\star\in\{\mathrm{in},\mathrm{fin}\}$. We claim that $\#D_t^\star\le 1$: Suppose not, and say that $k_1<k_2\in D_t^\star$. Then since $w(f_t)=m_{2j_0+1}$ there are $u_i\in\mathcal{B}_{k_i}^{\mathrm{unc}}$ and $s_i\in\mathcal{B}_{k_i,u_i}^\star$ $(i=1,2)$ such that $u_1,u_2\precneqq t\preceq s_1,s_2$. If $\star=\mathrm{in}$, then since $\mathrm{ran}\,f_t\subseteq\mathrm{ran}\,f_{s(k_1,u_1)}$, it follows that $\mathrm{ran}\,f_t<\mathrm{ran}\,w_{k_2}$, and since $\mathrm{ran}\,f_{s_2}\subseteq\mathrm{ran}\,f_t$, we obtain that $\mathrm{ran}\,f_{s_2}\cap\mathrm{ran}\,w_{k_2}=\emptyset$, contradicting the fact that

$s_2 \in \mathcal{B}^{\mathrm{in}}_{k_2,u_2}$. If $\star = \mathrm{fin}$, in a similar manner we obtain that $\operatorname{ran} w_{k_1} \cap \operatorname{ran} f_{s_1} = \emptyset$, a contradiction. Hence, either $G^\star_t = 0$, or $G^\star_t = (1/m_{2j_0+1})e^*_k$, certainly in $W(T_0)$ considered as sub-convex combinations.

Suppose now that $w(f_t) \neq m_{2j_0+1}$. There are three subcases to consider: If $w(f_t) > m_{2j_0+1}$, then t is non-relevant, hence a catcher node, and $G^\star_t = \sum_{k\in E^\star_t} g^\star_{k,t} = 0$. If $w(f_t) = m_{2j}$ with $j \leq j_0$, then the inductive hypothesis gives that $G^{\mathrm{cnd}}_t \in 3W(J_{T_0})$ (since $E^{\mathrm{cnd}}_t = \emptyset$). Fix $\star \in \{\mathrm{in}, \mathrm{fin}\}$. Observe that for every $k \in E^\star_t$ there is $s \in S_t$ such that $\operatorname{ran} f_s \subseteq \operatorname{ran} w_k$, in which case $D^\star_s = \emptyset$, and so $\# E^\star_t + \#\{s \in S_t \,:\, G^{\mathrm{cnd}}_s \neq 0\} \leq \# S_t$, and then, $G^\star_t = (1/w(f_t))(\sum_{k\in E^\star_t} e^*_k + \sum_{s\in S_t} G^\star_s) \in W(T_0)$.

If $w(f_t) = m_{2j+1} < m_{2j_0+1}$, then using that there is at most one immediate successor $s(t)$ of t which is relevant we obtain that either $G^{\mathrm{cnd}}_t = 0$, or $G^{\mathrm{cnd}}_t = (1/m_{2j})G^{\mathrm{cnd}}_{s(t)}$, and for $\star \in \{\mathrm{in}, \mathrm{fin}\}$, either $G^\star_t = (1/m_{2j+1})e^*_k$, or $G^\star_t = (1/m_{2j+1})G^\star_{s(t)}$.

(e.5): $\|R^{\mathrm{at}}_t\|_\infty \leq 1/m_{2j_0+1}$ follows from Proposition 8.9, since this is so for the corresponding assignment of which R^{at}_t is a filtration. Suppose that $\star \in \{\mathrm{in}, \mathrm{fin}, \mathrm{cnd}\}$. The proof is by backwards induction over t. Again we concentrate in non-trivial cases. Suppose that f_t is of type I and t is relevant. Then if $w(f_t) = m_{2j}$ with $j \leq j_0$, then the desired result follows from the definition of the corresponding assignments, and inductive hypothesis. Suppose that $w(f_t) = m_{2j+1}$ with $j < j_0$. Then $R^\star_t = \sum_{k\in E^\star_t} r^\star_{k,t} + (1/w(f_t))\sum_{s\in S_t} R^\star_s$. By the definition of the assignments, $\|r^\star_{k,t}\|_\infty \leq 1/m_{2j_0+1}$ for every $k \in E^\star_t$. Observe that all $s \in S_t$, except possibly one, $s(t)$, are non-relevant and that $r^\star_{k,s} = \|f_s(w_k)\| e^*_k$ for every $k \in E^\star_s = D^\star_s$. Hence, for every $s \in S_t \setminus \{s(t)\}$, $\|(1/w(f_t))R^\star_s\|_\infty = \max\{(1/w(f_t))\|f_s(w_k)\| \,:\, k \in E^\star_s\} \leq 1/m_{2j+1}$; the last inequality follows from Proposition 8.13. By the inductive hypothesis $\|R^\star_{s(t)}\|_\infty \leq 1/m_{2j_0+1}$, so we are done.

Suppose that t is non-relevant. The case $w(f_t) > m_{2j_0+1}$ is immediate. Suppose that $w(f_t) = m_{2j}$ and $t \neq s(u)$, where u is the immediate predecessor of t (see the definition of relevant node). Notice that t is a catcher, so $E^\star_t = D^\star_t$, and $R^\star_t = \sum_{k\in E^\star_t} \|f_t(w_k)\| e^*_k$, as desired. □

We are now ready to finish the proof using the part 8.1.3 of the general theory above. Notice that for each $\star \in \{\mathrm{in}, \mathrm{fin}, \mathrm{cnd}, \mathrm{at}\}$, $\boldsymbol{\mathcal{B}}^\star \not\prec \boldsymbol{\mathcal{B}}$, and that the canonical assignments of $\boldsymbol{\mathcal{B}}^\star$ are coherent. Let $(h^\star_{k,t})_{t\in\mathcal{B}_k,k}$ be the assignments induced by the canonical $\boldsymbol{\mathcal{B}}^\star$-assignments $(f^\star_{k,t})_{t\in\mathcal{B}^\star_k,k}$, for $\star \in \{\mathrm{in}, \mathrm{fin}, \mathrm{cnd}, \mathrm{at}\}$.

Claim. *For every $t \in \mathcal{T}$, $H^{\mathrm{in}}_t + H^{\mathrm{fin}}_t + F^{\mathrm{cnd}}_t + F^{\mathrm{at}}_t = F^{\boldsymbol{\mathcal{B}}}_t$, the canonical assignment of $\boldsymbol{\mathcal{B}}$.*

Proof of Claim: We show that for every $t \in B_k$, $h^{\mathrm{in}}_{k,t} + h^{\mathrm{fin}}_{k,t} + h^{\mathrm{cnd}}_{k,t} + h^{\mathrm{at}}_{k,t} = f_t(w_k)e^*_k$. The only non trivial case is if $t \in \mathcal{B}^{\mathrm{unc}}_k$. Notice that since $B^\star_{k,t}$ is a maximal antichain for $w^\star_{k,t}$ and $(f_s)_{s\succeq t}$, we obtain that $h^\star_{k,t} = F^\star_{k,t} = f_t(w^\star_{k,t})e^*_k$. Hence, $f^{\mathrm{in}}_{k,t} + f^{\mathrm{fin}}_{k,t} = (f_t(w^{\mathrm{in}}_{k,t}) + f_t(w^{\mathrm{fin}}_{k,t}))e^*_k = f_t(w_k)e^*_k$, and $h^\star_{k,t} = 0$ for $\star \in \{\mathrm{cnd}, \mathrm{at}\}$. □

Finally, by Proposition 8.11, $H^\star_\emptyset = F^\star_\emptyset$, for $\star \in \{\mathrm{in}, \mathrm{fin}, \mathrm{cnd}, \mathrm{at}\}$. Hence,

$$
\begin{aligned}
|f(\sum_{k=1}^n b_k w_k)| =& |F^{\boldsymbol{\mathcal{B}}}_\emptyset(\sum_{k=1}^n b_k e_k)| \leq \sum_{\star\in\{\mathrm{in},\mathrm{fin},\mathrm{cnd},\mathrm{at}\}} |F^\star_\emptyset(\sum_{k\in D^\star_\emptyset} b_k e_k)| \leq \\
&\leq 24(5\|\sum_{k=1}^n b_k e_k\|_{J_{T_0}} + 4\|\sum_{k=1}^n b_k e_k\|_\infty) \leq 120\|\sum_{k=1}^n e_k\|_{J_{T_0}} + \varepsilon. \qquad (71)
\end{aligned}
$$

□

Corollary 8.15. *The natural isomorphism $F : \langle w_1, \dots, w_n\rangle \to \langle v_1, \dots, v_n\rangle$ defined by $F(w_i) = v_i$ satisfies that $\|F\| \le 1$ and $\|F^{-1}\| \le 120+\varepsilon$. Consequently, J_{T_0} is finite interval representable on the basis $(e_\alpha)_{\alpha<\omega_1}$ of $\mathfrak{X}_{\omega_1}$ with a constant $\boldsymbol{C} < 121$.*

Proof. Proposition 5.11 shows that $\|F\| \le 1$; the other inequality follows from Lemma 8.14. □

9. The unconditional counterpart

We produce a space $\mathfrak{X}^u_{\omega_1}$ which is the counterpart of $\mathfrak{X}_{\omega_1}$ in the frame of the spaces with an unconditional basis, as in [11]. This space is defined as was $\mathfrak{X}_{\omega_1}$ by a norming family of functionals $K^u_{\omega_1}$ satisfying *(1)-(4)* from Subsection 2.2, and in addition the following condition

(5) It is closed under the restriction of all functionals with odd weight to every subset of ω_1

Although $\mathfrak{X}^u_{\omega_1}$ belongs to the class of spaces with an unconditional basis its study uses the same tools used in the study of $\mathfrak{X}_{\omega_1}$. For example, given a bounded operator $T : \mathfrak{X}^u_{\omega_1} \to \mathfrak{X}^u_{\omega_1}$ the transfinite sequence $(d(Te_\gamma, \mathbb{R}e_\gamma))_{\gamma<\omega_1}$ belongs to $c_0(\omega_1)$, and the operator T is strictly singular if and only if the sequence $(\|Te_\gamma\|)_{\gamma<\omega_1}$ belongs to $c_0(\omega_1)$.

Remark 9.1. 1. The basic inequality (Lemma 4.4) still remains true provided that (18) holds for an arbitrary subset $E \subseteq [1, n]$, not only for intervals.
2. For every block sequence $(y_n)_n$ of $\mathfrak{X}^u_{\omega_1}$ and every j there is a $(6, j)$-exact pair (y, ϕ) with $y \in \langle y_n\rangle_n$ (indeed, what one locates first are $2-\ell_1^n$ averages.)

The next result is the corresponding analogue from [13].

Proposition 9.2. *Let $T : \mathfrak{X}^u_{\omega_1} \to \mathfrak{X}^u_{\omega_1}$ be bounded, and let $(x_n)_n$ be a RIS of $\mathfrak{X}^u_{\omega_1}$. For each n, let $B_n \cup C_n = \operatorname{supp} x_n$ be a partition. Then $\lim_{n\to\infty} C_nTB_nx_n = 0$.*

Proof. (Sketch) Assume not. Notice that since $(x_n)_n$ is a block sequence, so is $(C_nTB_nx_n)_n$. Going to a subsequence if needed we assume that $\inf_n \|C_nTB_nx_n\| \ge \varepsilon > 0$. Since for every $\phi \in K_{u,\omega_1}$, the restriction $A\phi \in K_{u,\omega_1}$ for every subset $E \subseteq \omega_1$, we have that the sequence $(B_nx_n)_n$ is also RIS. Now for each n, choose $f_n \in K_{u,\omega_1}$ such that $\operatorname{supp} f_n \subseteq C_n$ and $f_n(C_nTB_nx_n) \ge \varepsilon$. Let j be such that $\|T\| < m_{2j+1}\varepsilon$, and find appropriate $(2j_i)_{i=1}^{n_{2j+1}}$ such that

$$(\frac{m_{2j_1}}{n_{2j_1}} \sum_{k\in F_1} B_kx_k, \frac{1}{m_{2j_1}} \sum_{k\in F_1} f_k, \dots, \frac{m_{2j_{n_{2j+1}}}}{n_{2j_{n_{2j+1}}}} \sum_{k\in F_{n_{2j+1}}} B_kx_k, \frac{1}{m_{2j_{n_{2j+1}}}} \sum_{k\in F_{n_{2j+1}}} f_k) \tag{72}$$

is a $(0, j)$-dependent sequence, for $F_1 < \cdots < F_{n_{2j+1}}$, each $\#F_i = n_{2j_i}$. Then, $\|Tx\| \ge \varepsilon/m_{2j+1} > \|T\|\|x\|$ where $x = 1/n_{2j+1} \sum_{i=1}^{n_{2j+1}} (m_{2j_i}/n_{2j_i} \sum_{k\in F_i} B_kx_k)$, a contradiction. □

Proposition 9.3. *Let $T : \mathfrak{X}^u_{\omega_1} \to \mathfrak{X}^u_{\omega_1}$ be bounded such that for all $\alpha < \omega_1$, $e^*_\alpha Te_\alpha = 0$. Then $\lim_{n\to\infty} Tx_n = 0$ for every RIS $(x_n)_n$.*

Proof. For each n, let $A_n = \operatorname{supp} x_n$.

Claim. $\lim_{n\to\infty} A_nTx_n = 0$.

Proof of Claim: Notice that

$$A_nTx_n = \left\{ \begin{array}{ll} \frac{2L_n(2L_n-1)}{L_n^2}\frac{1}{\#P_n}\sum_{(B,C)\in P_n} BTCx_n & \text{if } \#A_n \text{ even} \\ \frac{2L_n(2L_n+1)((L_n+1)^2+1)}{(L_n+1)^2(L_n^2+1)}\frac{1}{\#P_n}\sum_{(B,C)\in P_n} BTCx_n & \text{if } \#A_n \text{ odd} \end{array} \right. \tag{73}$$

where L_n is the entire part of $\#A_n/2$, and

$$P_n = \left\{ \begin{array}{ll} \{(B,C)\,:\,B\cup C = A_n,\, B\cap C=\emptyset,\, \#B = \operatorname{supp} x_n/2\} & \text{if } \#A_n \text{ even} \\ \{(B,C)\,:\,B\cup C = A_n,\, B\cap C=\emptyset,\, |\#B-\#C| = 1\} & \text{if } \#A_n \text{ odd} \end{array} \right. \tag{74}$$

Hence, $A_nTx_n = (\lambda_n/\#P_n)\sum_{(B,C)\in P_n} BTCx_n$ with $1\le\lambda_n\le 4$. By Proposition 9.2, $A_nTx_n \to_n 0$, as desired. □

Now suppose that $\lim_{n\to\infty} Tx_n \neq 0$. W.l.o.g. we may assume that $(Tx_n)_n$ is a block sequence and with support disjoint from $(x_n)_n$ (let γ_0 be the minimal $\gamma<\omega_1$ such that there is some infinite A such that $\inf_{n\in A}\|P_\gamma Tx_n\|>0$; now, replacing T by $P_{\gamma_0}T$, and going to a subsequence $(x_n)_{n\in A}$ we may assume that $(Tx_n)_n$ is a block sequence. By the previous Claim we obtain that $A_nTx_n \to_n 0$, so we may assume that $(Tx_n)_n$ and $(x_n)_n$ are disjointly supported). Now it is easy to produce, for large enough j, a $(0,j)$-dependent sequence $(y_1,\phi_1,\dots,y_{n_{2j+1}},\phi_{n_{2j+1}})$ such that $\|T((1/n_{2j+1})\sum_i y_i)\| > \|T\|\|(1/n_{2j+1})\sum_i y_i\|$, a contradiction. □

In the same way one can show the following useful result.

Proposition 9.4. *For every $X\hookrightarrow\mathfrak{X}^u_{\omega_1}$ generated by a block sequence $(x_n)_n\subseteq\mathfrak{X}^u_\gamma$, every bounded $T:X\to\mathfrak{X}^u_{[\gamma,\omega_1)}$ is strictly singular. Indeed, $\lim_{n\to\infty} Ty_n = 0$ for every RIS $(y_n)_n$ in X.* □

Corollary 9.5. *For every $X\hookrightarrow\mathfrak{X}^u_{\omega_1}$ and every $\varepsilon>0$, there is some block sequence $(z_n)_n$ of $\mathfrak{X}^u_{\omega_1}$ and some Schauder basis $(x_n)_n\subseteq X$ such that $\|z_n-x_n\|\le\varepsilon$.*

Proof. Fix $X\hookrightarrow\mathfrak{X}^u_{\omega_1}$. By standard facts of transfinite block sequences (see Proposition 1.3) we can find some $\lambda<\omega_1$ a block sequence $(w_n)_n$ of $\mathfrak{X}^u_\lambda$, and a sequence $(y_n)_n\subseteq X$ such that $\sum_n\|P_\lambda y_n - z_n\|\le\varepsilon/2$. W.l.o.g. (going to a block subsequence if needed) we may assume that $(z_n)_n$ is a RIS. Consider $U:\overline{\langle w_n\rangle_n}\to\mathfrak{X}^u_{\omega_1}$ defined by $U(w_n)=P_{[\lambda,\omega_1)}x_n$. Since $P_\lambda|\overline{\langle y_n\rangle_n}$ is an isomorphism, U is bounded. By Proposition 9.4, U is strictly singular. Hence we can find a block subsequence $(z_n)_n$ of $(w_n)_n$ and the corresponding block subsequence $(x_n)_n$ of $(y_n)_n$ such that $\|z_n-x_n\|\le\varepsilon$. □

Corollary 9.6. *If $T:\mathfrak{X}^u_{\omega_1}\to\mathfrak{X}^u_{\omega_1}$ is bounded and for all α we have that $e^*_\alpha Te_\alpha=0$, then T is strictly singular.*

Proof. Let $X\hookrightarrow\mathfrak{X}^u_{\omega_1}$, and fix $\varepsilon>0$. Choose some RIS $(z_n)_n$ and some sequence $(x_n)_n\subseteq X$ such that $\sum_n\|z_n-x_n\|\le\varepsilon/\|T\|$. By Proposition 9.3, $\lim_{n\to\infty}Tx_n=0$. Hence we can find $x\in\langle x_n\rangle_n$ such that $\|Tx\|\le\varepsilon$. □

Corollary 9.7. *For any $T:\mathfrak{X}^u_{\omega_1}\to\mathfrak{X}^u_{\omega_1}$, there is some diagonal operator D_T such that $S=T-D_T$ is strictly singular, $Se_\alpha=0$ for all $\alpha<\omega_1$ and S has separable range.*

Proof. Let $D_T:\mathfrak{X}^u_{\omega_1}\to\mathfrak{X}^u_{\omega_1}$ be defined for $\alpha<\omega_1$ by $D_T(e_\alpha)=e^*_\alpha(Te_\alpha)e_\alpha$. D_T is bounded and by Corollary 9.6, $T-D_T$ is strictly singular. □

Corollary 9.8. *For any infinite $A \subseteq \omega_1$, the space $\mathfrak{X}_A^u$ is reflexive with an unconditional basis and*

$$\mathcal{L}(\mathfrak{X}_A^u) \cong \mathcal{D}(\mathfrak{X}_A^u) \oplus \mathcal{S}(\mathfrak{X}_A^u). \tag{75}$$

□

Here $\mathcal{D}(\mathfrak{X}_A^u)$ denotes the space of the diagonal operators and $\mathcal{S}(\mathfrak{X}_A^u)$ is the space of strictly singular operators S with separable range such that $e_\alpha^*(Se_\alpha) = 0$ for every $\alpha \in A$.

Corollary 9.9. *For any infinite $A \subseteq \omega_1$, $\mathfrak{X}_A^u$ is not isomorphic to a proper subspace of itself.*

Proof. Let $X \hookrightarrow \mathfrak{X}_A^u$, $T : \mathfrak{X}_A^u \to X$ be an isomorphism and let $U = i_{X,\mathfrak{X}_A^u} \circ T$ be a semi-Fredholm operator with $\alpha(U) = 0$. Then $U = D_U + S$, D_U diagonal such that $D_T(e_\alpha) = e_\alpha^*(Te_\alpha)e_\alpha$, and S strictly singular. Since D_U is a strictly singular perturbation of the semi-Fredholm operator U with $\alpha(U) = 0$, D_U is semi-Fredholm, and $\alpha(D_U) < \infty$. But $\mathrm{Ker} D_U = \overline{\langle \{e_\alpha : Te_\alpha = 0\} \rangle}$. So, $D_U \mathfrak{X}_A^u = \overline{\langle \{e_\alpha : Te_\alpha \neq 0\} \rangle}$ which has co-dimension equal to $\alpha(D_U)$, hence D_U and U are Fredholm with index 0. Since U is 1-1, this implies that $X = \mathfrak{X}_A^u$, as desired. □

Corollary 9.10. *Let A, B two infinite sets of countable ordinals such that $A \cap B$ is finite. Then every bounded operator $T : \mathfrak{X}_A^u \to \mathfrak{X}_B^u$ is strictly singular.* □

Corollary 9.11. *There is a nonseparable reflexive space X with an unconditional basis such that*
(a) X is not isomorphic to any of its proper subspaces.
(b) Every bounded linear operator $T : X \to X$ is of the form $D + S$ with D a diagonal operator and S a strictly singular operator with separable range.
(c) For every I_1, I_2 infinite disjoint subsets of ω_1 the spaces $\mathfrak{X}_{I_1}$, $\mathfrak{X}_{I_2}$ are totally incomparable.

Suppose now that in addition ϱ is universal.

Corollary 9.12. *For every interval I of ordinals, $(e_\alpha)_{\alpha \in I}$ is nearly subsymmetric. Moreover, for any two minimal intervals $I = [\alpha, \alpha + \omega)$, $J = [\beta, \beta + \omega)$, $\mathfrak{X}_I^u$ is an asymptotic version of $\mathfrak{X}_J^u$.*

So, if we consider the version of $\mathfrak{X}_{\omega_1}^u$ obtained by a universal ϱ-function then the unconditional basis $(e_\alpha)_{\alpha < \omega_1}$ is nearly subsymmetric and for any pair of disjoint minimal infinite intervals I_1, I_2 $\mathfrak{X}_{I_1}^u$ is an asymptotic version of $\mathfrak{X}_{I_2}^u$, while they are totally incomparable.

Proposition 9.13. *The unconditional counterpart $\mathfrak{X}_{\omega_1}^u$ is arbitrarily distortable.*

Proof. The norms $(\| \cdot \|_{u,j})_j$ arbitrarily distort the space $\mathfrak{X}_{\omega_1}^u$, since $(6, j)$-exact pairs exist in every block sequence and by Corollary 9.5 every subspace $X \hookrightarrow \mathfrak{X}_{\omega_1}^u$ "almost" contains a block sequence. □

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