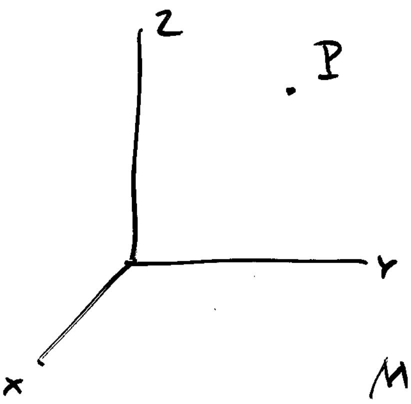
ΣΥΣΤΗΜΑΤΑ ΣΥΝΤΕΤΑΓΜΕΝΩΝ



κάθε σημείο P προσδιορίζεται

από (x, y, z)

Μπορούμε να θεωρήσουμε άλλο σύστημα συντεταγμένων: (q_1, q_2, q_3) έτσι ώστε:

$$\left.\begin{aligned} x &= x(q_1,q_2,q_3)\\ y &= y(q_1,q_2,q_3)\\ z &= z(q_1,q_2,q_3)\end{aligned}\right\} \Leftrightarrow \quad \begin{aligned} q_1 &= q_1(x,y,z)\\ q_2 &= q_2(x,y,z)\\ q_3 &= q_3(x,y,z)\end{aligned}$$

Απόσταση μεταξύ P, $P+dP$

$$P \downarrow (x,y,z) \qquad P+dP \downarrow (x+dx,\ y+dy,\ z+dz)$$

$$\boxed{ds^2 = dx^2 + dy^2 + dz^2}$$ Πυθαγόρειο Θεώρημα

$$dx = \frac{\partial x}{\partial q_1}dq_1 + \frac{\partial x}{\partial q_2}dq_2 + \frac{\partial x}{\partial q_3}dq_3 = \sum_i \frac{\partial x}{\partial q_i}dq_i$$

$$dy = \frac{\partial y}{\partial q_1}dq_1 + \frac{\partial y}{\partial q_2}dq_2 + \frac{\partial y}{\partial q_3}dq_3 = \sum_i \frac{\partial y}{\partial q_i}dq_i$$

$$dz = \frac{\partial z}{\partial q_1}dq_1 + \frac{\partial z}{\partial q_2}dq_2 + \frac{\partial z}{\partial q_3}dq_3 = \sum_i \frac{\partial z}{\partial q_i}dq_i$$

$$ds^2 = \left(\sum_i \frac{\partial x}{\partial q_i} dq_i\right)^2 + \left(\sum_i \frac{\partial y}{\partial q_i} dq_i\right)^2 + \left(\sum_i \frac{\partial z}{\partial q_i} dq_i\right)^2 =$$

$$= g_{11}\, dq_1^2 + g_{12}\, dq_2\, dq_1 + g_{13}\, dq_1\, dq_3 +$$

$$g_{21}\, dq_2\, dq_1 + g_{22}\, dq_2^2 + g_{23}\, dq_2\, dq_3 +$$

$$g_{31}\, dq_3\, dq_1 + g_{32}\, dq_3\, dq_2 + g_{33}\, dq_3^2 =$$

$$= \begin{pmatrix} dq_1 & dq_2 & dq_3 \end{pmatrix} \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} \begin{pmatrix} dq_1 \\ dq_2 \\ dq_3 \end{pmatrix}$$

$$= \sum_{i,j} \langle dq_i |\, g_{ij} \,| dq_j \rangle$$

$$g_{ij} = \frac{\partial x}{\partial q_i}\frac{\partial x}{\partial q_j} + \frac{\partial y}{\partial q_i}\frac{\partial y}{\partial q_j} + \frac{\partial z}{\partial q_i}\frac{\partial z}{\partial q_j} =$$

$$= \sum_{l} \frac{\partial x_l}{\partial q_i}\frac{\partial x_l}{\partial q_j}$$

$\boxed{g_{ij} \quad \text{μετρική}}$

ΟΡΘΟΓΩΝΙΑ ΣΥΣΤΗΜΑΤΑ ΣΥΝΤΕΤΑΓΜΕΝΩΝ

$$g_{ij} = 0 \qquad i \neq j$$

$$g_{ii} = h_i$$

$$g = \begin{pmatrix} h_1^2 & & 0 \\ & h_2^2 & \\ 0 & & h_3^2 \end{pmatrix}$$

$$ds^2 = g_{11} \cdot dq_1^2 + g_{22}\, dq_2^2 + g_{33}\, dq_3^2 =$$

$$= (h_1 dq_1)^2 + (h_2 dq_2)^2 + (h_3 dq_3)^2 =$$

$$= \sum_i (h_i dq_i)^2$$

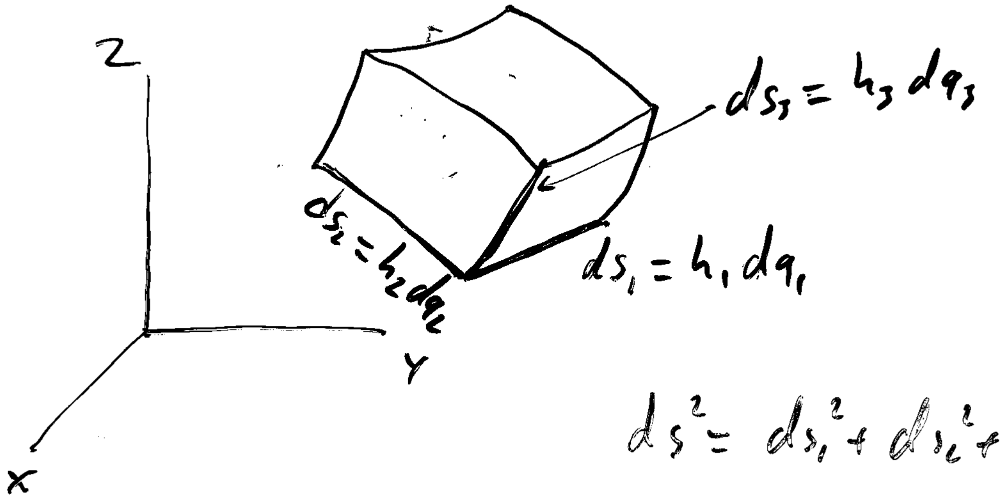
ΑΣΚΗΣΗ: Σφαιρικές συντεταγμένες:

$$q_1 = r, \quad q_2 = \theta \quad q_3 = \varphi$$

$$x = r \sin\theta \cos\varphi \qquad y = r \sin\theta \sin\varphi \qquad z = r\cos\theta$$

Υπολογίστε: 1) h_r, h_θ, h_φ

2) $ds^2 = \ldots$



$$ds^2 = ds_1^2 + ds_2^2 + ds_3^2$$

$$d^3V = dx\,dy\,dz = ds_1\,ds_2\,ds_3 =$$
$$= h_1 h_2 h_3\, dq_1\, dq_2\, dq_3$$

εμβαδόν πλευράς ij

$$d\sigma_{ij} = ds_i\, ds_j = h_i h_j\, dq_i\, dq_j$$

$$\nabla f = \hat{e}_1 \frac{\partial f}{\partial s_1} + \hat{e}_2 \frac{\partial f}{\partial s_2} + \hat{e}_3 \frac{\partial f}{\partial s_3} = \hat{e}_1 \frac{1}{h_1}\frac{\partial f}{\partial q_1} + \hat{e}_2 \frac{1}{h_2}\frac{\partial f}{\partial q_2} + \hat{e}_3 \frac{1}{h_3}\frac{\partial f}{\partial q_3}$$

$$\bar{\nabla}\cdot\bar{V} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}\left(h_2 h_3 V_1\right) + \frac{\partial}{\partial q_2}\left(h_1 h_3 V_2\right) + \frac{\partial}{\partial q_3}\left(h_1 h_2 V_3\right)\right]$$

$$\nabla^2 \psi = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial \psi}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_1 h_3}{h_2}\frac{\partial \psi}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial \psi}{\partial q_3}\right)\right]$$

$$\bar{\nabla}\times\bar{V} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} \hat{e}_1 h_1 & \hat{e}_2 h_2 & \hat{e}_3 h_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1 V_1 & h_2 V_2 & h_3 V_3 \end{vmatrix}$$

ΕΙΔΙΚΑ ΟΡΘΟΓΩΝΙΑ ΣΥΣΤΗΜΑΤΑ.

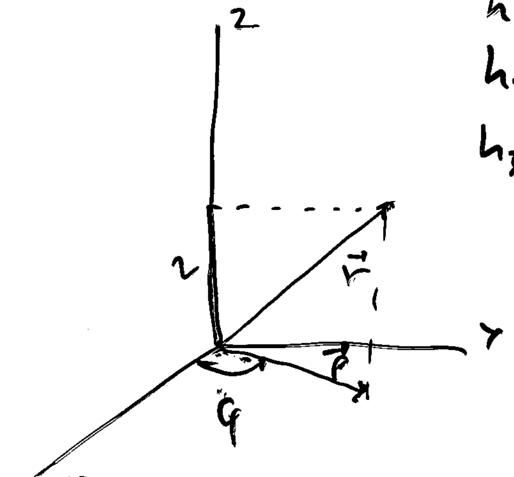
1) ΚΑΡΤΕΣΙΑΝΟ:

$$q_1 = x, \quad q_2 = y, \quad q_3 = z$$

$$h_1 = 1, \quad h_2 = 1 \quad h_3 = 1$$

2) ΚΥΛΙΝΔΡΙΚΟ:

$$\left.\begin{aligned} x &= \rho \cos\varphi \\ y &= \rho \sin\varphi \\ z &= z \end{aligned}\right\} \Longleftrightarrow \begin{aligned} \rho &= \sqrt{x^2+y^2} \\ \varphi &= \tan^{-1}\left(\frac{y}{x}\right) \\ z &= z \end{aligned}$$



$$h_1 = h_\rho = 1$$
$$h_2 = h_\varphi = \rho$$
$$h_3 = h_z = 1$$

$$d\vec{r} = \hat{\rho}\, ds_\rho + \hat{\varphi}\, ds_\varphi + \hat{z}\, ds_z$$

$$= \hat{\rho}\, d\rho + \hat{\varphi}\, \rho\, d\varphi + \hat{z}\, dz$$

$$ds^2 = d\rho^2 + \rho^2 d\varphi^2 + dz^2$$

$$\nabla f = \hat{\rho}\,\frac{\partial f}{\partial \rho} + \hat{\varphi}\,\frac{1}{\rho}\frac{\partial f}{\partial \varphi} + \hat{z}\,\frac{\partial f}{\partial z}$$

$$\bar{\nabla}\cdot\bar{v} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho v_\rho\right) + \frac{1}{\rho}\frac{\partial v_\varphi}{\partial \varphi} + \frac{\partial v}{\partial z}$$

$$\nabla^2 f = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\bar{\nabla}\times\bar{v} = \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\varphi} & \hat{z} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ v_\rho & \rho v_\varphi & v_z \end{vmatrix}$$

Άσκηση: Η εξίσωση Navier-Stokes στην υδροδυναμική περιέχει τον μη γραμμικό όρο:

$$\vec{A} = \vec{\nabla}\times\left[\vec{v}\times\left(\vec{\nabla}\times\vec{v}\right)\right].$$

$\vec{v}$ είναι η ταχύτητα του ρευστού.

Για ρευστό που κινείται σε κυλινδρικό σωλήνα στην κατεύθυνση z είναι:

$$\vec{v} = \hat{z}\,v(\rho)$$

Αποδείξτε ότι $\underline{\vec{A} = 0}$

Time derivatives of the unit vectors

We will also have many uses for the time derivatives of the unit vectors expressed in cylindrical coordinates:

$$\dot{\hat{\rho}} = \frac{\partial\hat{\rho}}{\partial\rho}\dot{\rho} + \frac{\partial\hat{\rho}}{\partial\phi}\dot{\phi} + \frac{\partial\hat{\rho}}{\partial z}\dot{z} = \hat{\phi}\dot{\phi}$$

$$\dot{\hat{\phi}} = \frac{\partial\hat{\phi}}{\partial\rho}\dot{\rho} + \frac{\partial\hat{\phi}}{\partial\phi}\dot{\phi} + \frac{\partial\hat{\phi}}{\partial z}\dot{z} = -\hat{\rho}\dot{\phi}$$

$$\dot{\hat{z}} = \frac{\partial\hat{z}}{\partial\rho}\dot{\rho} + \frac{\partial\hat{z}}{\partial\phi}\dot{\phi} + \frac{\partial\hat{z}}{\partial z}\dot{z} = 0$$

Velocity and Acceleration

The velocity and acceleration of a particle may be expressed in cylindrical coordinates by taking into account the associated rates of change in the unit vectors:

$$\bar{v} = \dot{\bar{r}} = \dot{\hat{\rho}}\rho + \hat{\rho}\dot{\rho} + \dot{\hat{z}}z + \hat{z}\dot{z} = \hat{\rho}\dot{\rho} + \hat{\phi}\rho\dot{\phi} + \hat{z}\dot{z}$$

$$\boxed{\bar{v} = \hat{\rho}\dot{\rho} + \hat{\phi}\rho\dot{\phi} + \hat{z}\dot{z}}$$

$$\bar{a} = \dot{\bar{v}} = \dot{\hat{\rho}}\dot{\rho} + \hat{\rho}\ddot{\rho} + \dot{\hat{\phi}}\rho\dot{\phi} + \hat{\phi}\dot{\rho}\dot{\phi} + \hat{\phi}\rho\ddot{\phi} + \dot{\hat{z}}\dot{z} + \hat{z}\ddot{z}$$
$$= \hat{\phi}\dot{\phi}\dot{\rho} + \hat{\rho}\ddot{\rho} - \hat{\rho}\rho\dot{\phi}^2 + \hat{\phi}\dot{\rho}\dot{\phi} + \hat{\phi}\rho\ddot{\phi} + \hat{z}\ddot{z}$$

$$\boxed{\bar{a} = \hat{\rho}\left(\ddot{\rho} - \rho\dot{\phi}^2\right) + \hat{\phi}\left(\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi}\right) + \hat{z}\ddot{z}}$$

The del operator from the definition of the gradient

Any (static) scalar field u may be considered to be a function of the cylindrical coordinates ρ, ϕ, and z. The value of u changes by an infinitesimal amount du when the point of observation is changed by $d\bar{r}$. That change may be determined from the partial derivatives as

$$du = \frac{\partial u}{\partial\rho}d\rho + \frac{\partial u}{\partial\phi}d\phi + \frac{\partial u}{\partial z}dz\,.$$

But we also define the gradient in such a way as to obtain the result

$$du = \bar{\nabla}u \cdot d\bar{r}$$

Therefore,

$$\frac{\partial u}{\partial\rho}d\rho + \frac{\partial u}{\partial\phi}d\phi + \frac{\partial u}{\partial z}dz = \bar{\nabla}u \cdot d\bar{r}$$

or, in cylindrical coordinates,

$$\frac{\partial u}{\partial\rho}d\rho + \frac{\partial u}{\partial\phi}d\phi + \frac{\partial u}{\partial z}dz = \left(\bar{\nabla}u\right)_\rho d\rho + \left(\bar{\nabla}u\right)_\phi \rho\, d\phi + \left(\bar{\nabla}u\right)_z dz$$

and we demand that this hold for any choice of $d\rho$, $d\phi$ and dz. Thus,

$$\left(\bar{\nabla}u\right)_\rho = \frac{\partial u}{\partial\rho},\quad \left(\bar{\nabla}u\right)_\phi = \frac{1}{\rho}\frac{\partial u}{\partial\phi},\quad \left(\bar{\nabla}u\right)_z = \frac{\partial u}{\partial z},$$

from which we find

$$\boxed{\bar{\nabla} = \hat{\rho}\frac{\partial}{\partial\rho} + \frac{\hat{\phi}}{\rho}\frac{\partial}{\partial\phi} + \hat{z}\frac{\partial}{\partial z}}$$

Divergence

The divergence $\bar{\nabla} \cdot \bar{A}$ is carried out taking into account, once again, that the unit vectors themselves are functions of the coordinates. Thus, we have

$$\bar{\nabla} \cdot \bar{A} = \left(\hat{\rho} \frac{\partial}{\partial \rho} + \frac{\hat{\phi}}{\rho} \frac{\partial}{\partial \phi} + \hat{z} \frac{\partial}{\partial z} \right) \cdot \left(A_\rho \hat{\rho} + A_\phi \hat{\phi} + A_z \hat{z} \right)$$

where the derivatives must be taken *before* the dot product so that

$$\begin{aligned}
\bar{\nabla} \cdot \bar{A} &= \left(\hat{\rho} \frac{\partial}{\partial \rho} + \frac{\hat{\phi}}{\rho} \frac{\partial}{\partial \phi} + \hat{z} \frac{\partial}{\partial z} \right) \cdot \bar{A} \\
&= \hat{\rho} \cdot \frac{\partial \bar{A}}{\partial \rho} + \frac{\hat{\phi}}{\rho} \cdot \frac{\partial \bar{A}}{\partial \phi} + \hat{z} \cdot \frac{\partial \bar{A}}{\partial z} \\
&= \hat{\rho} \cdot \left(\frac{\partial A_\rho}{\partial \rho} \hat{\rho} + \frac{\partial A_\phi}{\partial \rho} \hat{\phi} + \frac{\partial A_z}{\partial \rho} \hat{z} + A_\rho \frac{\partial \hat{\rho}}{\partial \rho} + A_\phi \frac{\partial \hat{\phi}}{\partial \rho} + A_z \frac{\partial \hat{z}}{\partial \rho} \right) \\
&\quad + \frac{\hat{\phi}}{\rho} \cdot \left(\frac{\partial A_\rho}{\partial \phi} \hat{\rho} + \frac{\partial A_\phi}{\partial \phi} \hat{\phi} + \frac{\partial A_z}{\partial \phi} \hat{z} + A_\rho \frac{\partial \hat{\rho}}{\partial \phi} + A_\phi \frac{\partial \hat{\phi}}{\partial \phi} + A_z \frac{\partial \hat{z}}{\partial \phi} \right) \\
&\quad + \hat{z} \cdot \left(\frac{\partial A_\rho}{\partial z} \hat{\rho} + \frac{\partial A_\phi}{\partial z} \hat{\phi} + \frac{\partial A_z}{\partial z} \hat{z} + A_\rho \frac{\partial \hat{\rho}}{\partial z} + A_\phi \frac{\partial \hat{\phi}}{\partial z} + A_z \frac{\partial \hat{z}}{\partial z} \right)
\end{aligned}$$

With the help of the partial derivatives previously obtained, we find

$$\begin{aligned}
\bar{\nabla} \cdot \bar{A} &= \hat{\rho} \cdot \left(\frac{\partial A_\rho}{\partial \rho} \hat{\rho} + \frac{\partial A_\phi}{\partial \rho} \hat{\phi} + \frac{\partial A_z}{\partial \rho} \hat{z} + 0 + 0 + 0 \right) \\
&\quad + \frac{\hat{\phi}}{\rho} \cdot \left(\frac{\partial A_\rho}{\partial \phi} \hat{\rho} + \frac{\partial A_\phi}{\partial \phi} \hat{\phi} + \frac{\partial A_z}{\partial \phi} \hat{z} + A_\rho \hat{\phi} - A_\phi \hat{\rho} + 0 \right) \\
&\quad + \hat{z} \cdot \left(\frac{\partial A_\rho}{\partial z} \hat{\rho} + \frac{\partial A_\phi}{\partial z} \hat{\phi} + \frac{\partial A_z}{\partial z} \hat{z} + 0 + 0 + 0 \right) \\
&= \left(\frac{\partial A_\rho}{\partial \rho} \right) + \left(\frac{1}{\rho} \frac{\partial A_\phi}{\partial \phi} + \frac{A_\rho}{\rho} \right) + \left(\frac{\partial A_z}{\partial z} \right) \\
&= \left(\frac{\partial A_\rho}{\partial \rho} + \frac{A_\rho}{\rho} \right) + \frac{1}{\rho} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}
\end{aligned}$$

$$\boxed{\bar{\nabla} \cdot \bar{A} = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(A_\rho \rho \right) + \frac{1}{\rho} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}}$$

Curl

The curl $\bar{\nabla} \times \bar{A}$ is also carried out taking into account that the unit vectors themselves are functions of the coordinates. Thus, we have

$$\bar{\nabla} \times \bar{A} = \left(\hat{\rho} \frac{\partial}{\partial \rho} + \frac{\hat{\phi}}{\rho} \frac{\partial}{\partial \phi} + \hat{z} \frac{\partial}{\partial z} \right) \times \left(A_\rho \hat{\rho} + A_\phi \hat{\phi} + A_z \hat{z} \right)$$

where the derivatives must be taken *before* the cross product so that

$$\begin{aligned}
\bar{\nabla} \times \bar{A} &= \left(\hat{\rho} \frac{\partial}{\partial \rho} + \frac{\hat{\phi}}{\rho} \frac{\partial}{\partial \phi} + \hat{z} \frac{\partial}{\partial z} \right) \times \bar{A} \\
&= \hat{\rho} \times \frac{\partial \bar{A}}{\partial \rho} + \frac{\hat{\phi}}{\rho} \times \frac{\partial \bar{A}}{\partial \phi} + \hat{z} \times \frac{\partial \bar{A}}{\partial z} \\
&= \hat{\rho} \times \left(\frac{\partial A_\rho}{\partial \rho} \hat{\rho} + \frac{\partial A_\phi}{\partial \rho} \hat{\phi} + \frac{\partial A_z}{\partial \rho} \hat{z} + A_\rho \frac{\partial \hat{\rho}}{\partial \rho} + A_\phi \frac{\partial \hat{\phi}}{\partial \rho} + A_z \frac{\partial \hat{z}}{\partial \rho} \right) \\
&\quad + \frac{\hat{\phi}}{\rho} \times \left(\frac{\partial A_\rho}{\partial \phi} \hat{\rho} + \frac{\partial A_\phi}{\partial \phi} \hat{\phi} + \frac{\partial A_z}{\partial \phi} \hat{z} + A_\rho \frac{\partial \hat{\rho}}{\partial \phi} + A_\phi \frac{\partial \hat{\phi}}{\partial \phi} + A_z \frac{\partial \hat{z}}{\partial \phi} \right) \\
&\quad + \hat{z} \times \left(\frac{\partial A_\rho}{\partial z} \hat{\rho} + \frac{\partial A_\phi}{\partial z} \hat{\phi} + \frac{\partial A_z}{\partial z} \hat{z} + A_\rho \frac{\partial \hat{\rho}}{\partial z} + A_\phi \frac{\partial \hat{\phi}}{\partial z} + A_z \frac{\partial \hat{z}}{\partial z} \right)
\end{aligned}$$

With the help of the partial derivatives previously obtained, we find

$$\begin{aligned}
\bar{\nabla} \times \bar{A} &= \hat{\rho} \times \left(\frac{\partial A_\rho}{\partial \rho} \hat{\rho} + \frac{\partial A_\phi}{\partial \rho} \hat{\phi} + \frac{\partial A_z}{\partial \rho} \hat{z} + 0 + 0 + 0 \right) \\
&\quad + \frac{\hat{\phi}}{\rho} \times \left(\frac{\partial A_\rho}{\partial \phi} \hat{\rho} + \frac{\partial A_\phi}{\partial \phi} \hat{\phi} + \frac{\partial A_z}{\partial \phi} \hat{z} + A_\rho \hat{\phi} - A_\phi \hat{\rho} + 0 \right) \\
&\quad + \hat{z} \times \left(\frac{\partial A_\rho}{\partial z} \hat{\rho} + \frac{\partial A_\phi}{\partial z} \hat{\phi} + \frac{\partial A_z}{\partial z} \hat{z} + 0 + 0 + 0 \right) \\
&= \left(\frac{\partial A_\phi}{\partial \rho} \hat{z} - \frac{\partial A_z}{\partial \rho} \hat{\phi} \right) + \left(-\frac{1}{\rho} \frac{\partial A_\rho}{\partial \phi} \hat{z} + \frac{1}{\rho} \frac{\partial A_z}{\partial \phi} \hat{\rho} + \frac{A_\phi}{\rho} \hat{z} \right) \\
&\quad + \left(\frac{\partial A_\rho}{\partial z} \hat{\phi} - \frac{\partial A_\phi}{\partial z} \hat{\rho} \right) \\
&= \hat{\rho} \left(\frac{1}{\rho} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right) + \hat{\phi} \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho} \right) + \hat{z} \left(\frac{\partial A_\phi}{\partial \rho} + \frac{A_\phi}{\rho} - \frac{1}{\rho} \frac{\partial A_\rho}{\partial \phi} \right)
\end{aligned}$$

$$\boxed{\bar{\nabla} \times \bar{A} = \hat{\rho} \left(\frac{1}{\rho} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right) + \hat{\phi} \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho} \right) + \hat{z} \left(\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(A_\phi \rho \right) - \frac{1}{\rho} \frac{\partial A_\rho}{\partial \phi} \right)}$$

Laplacian

The Laplacian is a scalar operator that can be determined from its definition as

$$\nabla^2 u = \vec{\nabla}\cdot\left(\vec{\nabla}u\right) = \left(\hat{\rho}\frac{\partial}{\partial\rho} + \frac{\hat{\phi}}{\rho}\frac{\partial}{\partial\phi} + \hat{z}\frac{\partial}{\partial z}\right)\cdot\left(\hat{\rho}\frac{\partial u}{\partial\rho} + \frac{\hat{\phi}}{\rho}\frac{\partial u}{\partial\phi} + \hat{z}\frac{\partial u}{\partial z}\right)$$

$$= \hat{\rho}\cdot\frac{\partial}{\partial\rho}\left(\hat{\rho}\frac{\partial u}{\partial\rho} + \frac{\hat{\phi}}{\rho}\frac{\partial u}{\partial\phi} + \hat{z}\frac{\partial u}{\partial z}\right)$$

$$+ \frac{\hat{\phi}}{\rho}\cdot\frac{\partial}{\partial\phi}\left(\hat{\rho}\frac{\partial u}{\partial\rho} + \frac{\hat{\phi}}{\rho}\frac{\partial u}{\partial\phi} + \hat{z}\frac{\partial u}{\partial z}\right)$$

$$+ \hat{z}\cdot\frac{\partial}{\partial z}\left(\hat{\rho}\frac{\partial u}{\partial\rho} + \frac{\hat{\phi}}{\rho}\frac{\partial u}{\partial\phi} + \hat{z}\frac{\partial u}{\partial z}\right)$$

With the help of the partial derivatives previously obtained, we find

$$\nabla^2 u = \hat{\rho}\cdot\left(\hat{\rho}\frac{\partial^2 u}{\partial\rho^2} - \frac{\hat{\phi}}{\rho^2}\frac{\partial u}{\partial\phi} + \frac{\hat{\phi}}{\rho}\frac{\partial^2 u}{\partial\phi\partial\rho} + \hat{z}\frac{\partial^2 u}{\partial z\partial\rho}\right)$$

$$+ \frac{\hat{\phi}}{\rho}\cdot\left(\hat{\phi}\frac{\partial u}{\partial\rho} + \hat{\rho}\frac{\partial^2 u}{\partial\rho\partial\phi} - \frac{\hat{\rho}}{\rho}\frac{\partial u}{\partial\phi} + \frac{\hat{\phi}}{\rho}\frac{\partial^2 u}{\partial\phi^2} + \hat{z}\frac{\partial^2 u}{\partial z\partial\phi}\right)$$

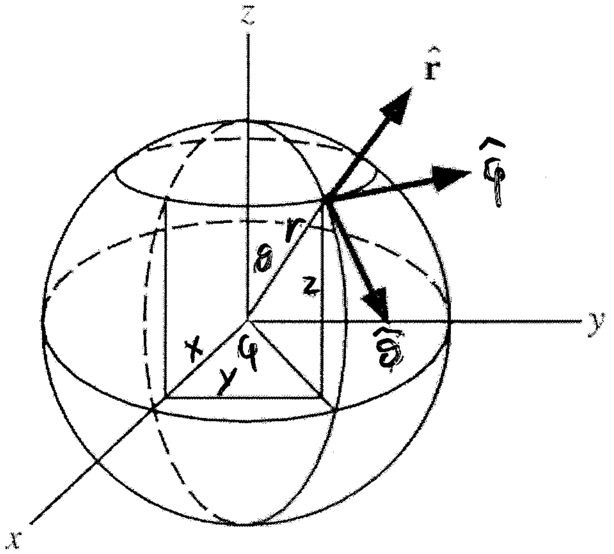
$$+ \hat{z}\cdot\left(\hat{\rho}\frac{\partial^2 u}{\partial\rho\partial z} + \frac{\hat{\phi}}{\rho}\frac{\partial^2 u}{\partial\phi\partial z} + \hat{z}\frac{\partial^2 u}{\partial z^2}\right)$$

$$= \frac{\partial^2 u}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial\phi^2} + \frac{\partial^2 u}{\partial z^2}$$

$$= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial\phi^2} + \frac{\partial^2 u}{\partial z^2}$$

Thus, the Laplacian operator can be written as

$$\boxed{\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}}$$



3) ΣΦΑΙΡΙΚΕΣ:

$$x = r\sin\theta\cos\varphi$$
$$y = r\sin\theta\sin\varphi$$
$$z = r\cos\theta$$

$$\Rightarrow \quad r = \sqrt{x^2+y^2+z^2}, \quad \theta = \cos^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}}, \quad \varphi = \tan^{-1}\frac{y}{x}$$

$h_1 = h_r = 1$

$h_2 = h_\theta = r$

$h_3 = h_\varphi = r\sin\theta$

$$d\vec{r} = \hat{r}\,dr + \hat{\theta}\,r\,d\theta + \hat{\varphi}\,r\sin\theta\,d\varphi$$

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\,d\varphi^2$$

$r = \text{const}$ $\quad dA = d\sigma_{\theta\varphi} = h_\theta h_\varphi\,d\theta\,d\varphi = r^2\sin\theta\,d\theta\,d\varphi$

$$d^3V = h_1 h_2 h_3\,dq_1\,dq_2\,dq_3 = r^2\sin\theta\,dr\,d\theta\,d\varphi$$

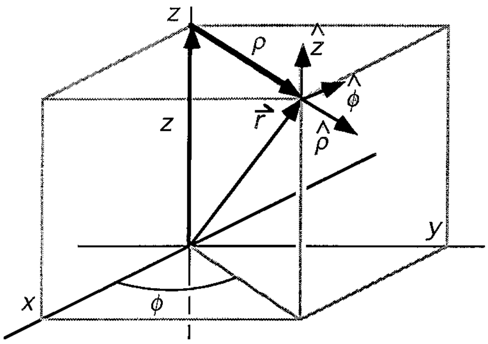
$$\nabla f = \hat{r}\,\frac{\partial f}{\partial r} + \hat{\theta}\,\frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{\varphi}\,\frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}$$

$$\nabla\cdot V = \frac{1}{r^2\sin\theta}\left[\sin\theta\,\frac{\partial}{\partial r}\left(r^2 V_r\right) + r\,\frac{\partial}{\partial \theta}\left(\sin\theta\, V_\theta\right) + r\,\frac{\partial V}{\partial \varphi}\right]$$

$$\nabla^2 f = \frac{1}{r^2\sin\theta}\left[\sin\theta\,\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta\,\frac{\partial f}{\partial \theta}\right) + \frac{1}{\sin\theta}\,\frac{\partial^2 f}{\partial \varphi^2}\right]$$

$$\bar{\nabla}\times\bar{v} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\,\hat{\theta} & r\sin\theta\,\hat{\varphi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ V_r & r\,V_\theta & r\sin\theta\,V_\varphi \end{vmatrix}$$

Cylindrical Coordinates



Transforms

The forward and reverse coordinate transformations are

$$\rho = \sqrt{x^2 + y^2} \qquad x = \rho\cos\phi$$
$$\phi = \arctan(y, x) \qquad y = \rho\sin\phi$$
$$z = z \qquad z = z$$

where we *formally* take advantage of the *two argument* arctan function to eliminate quadrant confusion.

Unit Vectors

The unit vectors in the cylindrical coordinate system are functions of position. It is convenient to express them in terms of the *cylindrical* coordinates and the unit vectors of the *rectangular* coordinate system which are *not* themselves functions of position.

$$\hat{\rho} = \frac{\bar{\rho}}{\rho} = \frac{x\hat{x} + y\hat{y}}{\rho} = \hat{x}\cos\phi + \hat{y}\sin\phi$$

$$\hat{\phi} = \hat{z} \times \hat{\rho} = -\hat{x}\sin\phi + \hat{y}\cos\phi$$

$$\hat{z} = \hat{z}$$

Variations of unit vectors with the coordinates

Using the expressions obtained above it is easy to derive the following handy relationships:

$$\frac{\partial\hat{\rho}}{\partial\rho} = 0 \qquad \frac{\partial\hat{\phi}}{\partial\rho} = 0 \qquad \frac{\partial\hat{z}}{\partial\rho} = 0$$

$$\frac{\partial\hat{\rho}}{\partial\phi} = -\hat{x}\sin\phi + \hat{y}\cos\phi = \hat{\phi} \qquad \frac{\partial\hat{\phi}}{\partial\phi} = -\hat{x}\cos\phi - \hat{y}\sin\phi = -\hat{\rho} \qquad \frac{\partial\hat{z}}{\partial\phi} = 0$$

$$\frac{\partial\hat{\rho}}{\partial z} = 0 \qquad \frac{\partial\hat{\phi}}{\partial z} = 0 \qquad \frac{\partial\hat{z}}{\partial z} = 0$$

Path increment

We will have many uses for the path increment $d\bar{r}$ expressed in cylindrical coordinates:

$$\begin{aligned}
d\bar{r} &= d(\rho\hat{\rho} + z\hat{z}) = \hat{\rho}d\rho + \rho d\hat{\rho} + \hat{z}dz + zd\hat{z} \\
&= \hat{\rho}d\rho + \rho\left(\frac{\partial\hat{\rho}}{\partial\rho}d\rho + \frac{\partial\hat{\rho}}{\partial\phi}d\phi + \frac{\partial\hat{\rho}}{\partial z}dz\right) + \hat{z}dz + z\left(\frac{\partial\hat{z}}{\partial\rho}d\rho + \frac{\partial\hat{z}}{\partial\phi}d\phi + \frac{\partial\hat{z}}{\partial z}dz\right) \\
&= \hat{\rho}d\rho + \hat{\phi}\rho d\phi + \hat{z}dz
\end{aligned}$$

Άσκηση 1:

Έστω δυναμ. πεδίο $\vec{F}$:

$$\vec{F} = \hat{e}_r \frac{\cos\theta}{r^3} + \hat{e}_\theta \frac{\sin\theta}{r^3}$$

A.) Είναι το $\vec{F}$ αστρόβιλο; ($\vec{\nabla}\times\vec{F} = 0$?)

B.) Αν ναι, ποιό είναι το δυναμικό του U: ($\vec{F} = -\vec{\nabla}U$)

Γ.) Έχει πηγές; ($\vec{\nabla}\cdot\vec{F} = ;$)

Άσκηση 2.

Έστω μαγνητικό πεδίο $\vec{B}$:

$$\vec{B} = \frac{g}{4\pi}\frac{\vec{r}}{r^3}$$

A) Είναι το $\vec{B}$ αστρόβιλο;

B) Έχει πηγές; ($\vec{\nabla}\cdot\vec{B} = ;$)