ΔΙΑΦΟΡΙΚΟΙ ΤΕΛΕΣΤΕΣ

$$\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right)$$

$$(\vec{\nabla})_i = \frac{\partial}{\partial x_i} \equiv \partial_i$$

1) Κλίση:

$$\vec{\nabla} f = \text{grad}\, f = (\partial_1 f, \partial_2 f, \partial_3 f)$$

<u>διάνυσμα</u>

2) Απόκλιση:

$$\vec{\nabla} \cdot \vec{h} = \text{div}\, \vec{h} = \partial_1 h_1 + \partial_2 h_2 + \partial_3 h_3$$

<u>βαθμωτό</u>

3) Περιστροφή

$$\vec{\nabla} \times \vec{A} = \text{curl}\, \vec{A} = (\partial_2 A_3 - \partial_3 A_2, \partial_3 A_1 - \partial_1 A_3, \partial_1 A_2 - \partial_2 A_1)$$

<u>διάνυσμα</u>

ΕΞΙΣΩΣΕΙΣ MAXWELL

$$\vec{\nabla} \cdot \vec{E} = \rho / \varepsilon_0$$

$$\vec{\nabla} \times \vec{E} = - \frac{\partial B}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{B} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} + \frac{1}{c^2 \epsilon_0} \vec{J}$$

ΠΑΡΑΓΩΓΟΙ 2ης ΤΑΞΗΣ:

α) $\vec{\nabla}\cdot(\vec{\nabla}T)$

β) $\vec{\nabla}\times(\vec{\nabla}T)$

γ) $\vec{\nabla}(\vec{\nabla}\vec{h})$

δ) $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{h})$

ε) $\vec{\nabla}\times(\vec{\nabla}\times\vec{h})$

a) $\vec{\nabla}\cdot(\vec{\nabla}T) = (\hat{e}_1\partial_1 + \hat{e}_2\partial_2 + \hat{e}_3\partial_3)(\hat{e}_1\partial_1 T + \hat{e}_2\partial_2 T + \hat{e}_3\partial_3 T)$

$= \partial_1^2 T + \partial_2^2 T + \partial_3^2 T$

Λαπλασιανή $= \nabla^2 = \dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_2^2} + \dfrac{\partial^2}{\partial x_3^2}$

$$\underline{\vec{\nabla}\cdot(\vec{\nabla}T) = \nabla^2 T}$$

$$\nabla^2\vec{h} \equiv (\nabla^2 h_1, \nabla^2 h_2, \nabla^2 h_3)$$

α) $\vec{\nabla} \times \vec{\nabla} T = \;?$

β) $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{u}) = \;?$

α) $(\vec{\nabla} \times \vec{\nabla} T)_i = \varepsilon_{ijk}\, \partial_j \partial_k T = 0$

β) $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{u}) = \partial_i \left(\varepsilon_{ijk}\, \partial_j u_k \right) = \varepsilon_{ijk}\, \partial_i \partial_j u_k = 0$

- η περιστροφή της κλίσης $= 0$
- απόκλιση της περιστροφής $= 0$

ΘΕΩΡΗΜΑ I:

Αν $\vec{\nabla} \times \vec{A} = 0 \;\Rightarrow\; \vec{A} = \nabla f$

ΘΕΩΡΗΜΑ II:

Αν $\vec{\nabla} \cdot \vec{D} = 0 \;\Rightarrow\; \vec{D} = \vec{\nabla} \times \vec{C}$

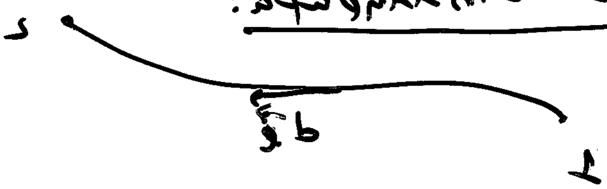
ΔΙΑΝΥΣΜΑΤΙΚΑ ΟΛΟΚΛΗΡΩΜΑΤΑ

$$\int_a^b g(x)\,dx$$

$$\int_a^b f'(x)\,dx = f(b)-f(a)$$

Ομοίως ορίζω τα <u>επικαμπύλια</u> ολοκληρώματα:

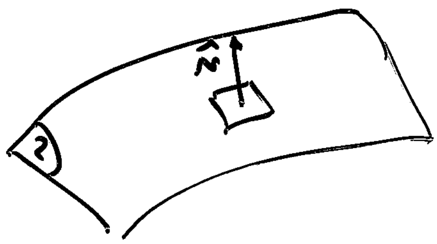
Επικαμπύλια ολοκληρώματα:



$$\int_1^2 \vec{A}\cdot d\vec{l}$$

Ισχύει: $\int_1^2 \vec{\nabla} f\cdot d\vec{l} = f(2)-f(1)$

Επιφανειακά ολοκληρώματα:

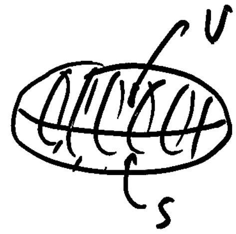


$$d\vec{S} = \hat{n}\,da$$

$$\underbrace{\int_S \vec{A}\cdot d\vec{S}}_{\text{ροή}} = \int_S \vec{A}\cdot\hat{n}\,da$$

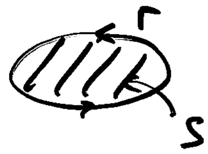
ΘΕΩΡΗΜΑ GAUSS:

$$\oint_S \vec{C}\cdot\hat{n}\,da = \int_V \vec{\nabla}\cdot\vec{C}\,dV$$



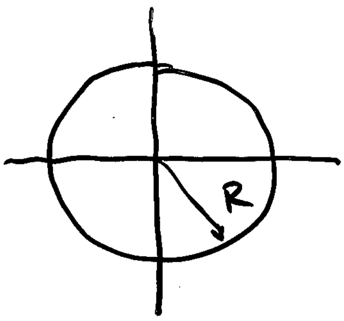
ΘΕΩΡΗΜΑ STOKES:

$$\oint_\Gamma \vec{C}\cdot d\vec{\ell} = \int_S (\nabla\times\vec{C})\cdot\hat{n}\,da$$



ΠΑΡΑΔΕΙΓΜΑ:

GAUSS: $\underline{\underline{S}}$: Σφαίρα ακτίνας R



$$\vec{C} = z^3\,\hat{e}_3$$

$$\oint_S \vec{C}\cdot\hat{n}\,da$$

C πάνω στη σφαίρα είναι

$$\vec{C} = R^3\cos^3\theta\,\hat{e}_3$$

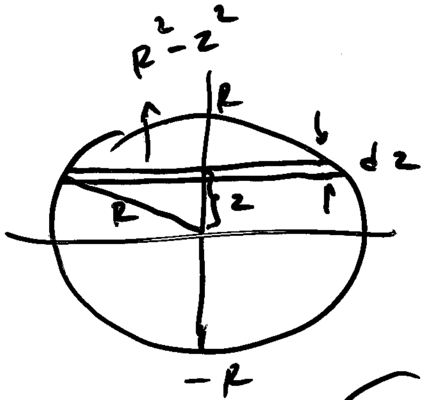
$$\hat{n} = \hat{r} \qquad \hat{r}\cdot\hat{e}_3 = \cos\theta$$

$$da = R^2 \sin\theta \, d\theta \, d\varphi$$

$$\oint \vec{C} \cdot \hat{n} \, da = \int_0^{\pi} \int_0^{2\pi} R^3 \cos^3\theta \; R^2 \sin\theta \, d\theta \, d\varphi$$

$$= 2\pi R^5 \int_0^{\pi} \cos^4\theta \sin\theta \, d\theta = \frac{4\pi R^5}{5}$$

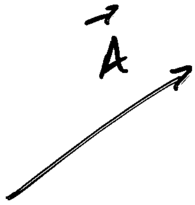
$$\vec{\nabla} \cdot \vec{C} = 3z^2$$



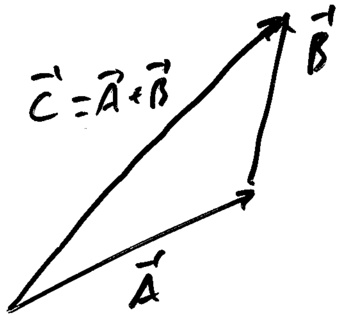
$$dV = \pi (R^2 - z^2) \, dz$$

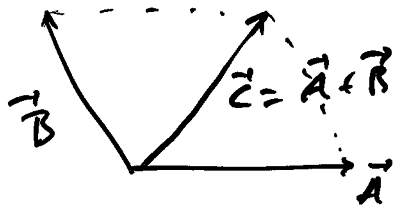
$$\int_V (\vec{\nabla} \cdot \vec{C}) \, dV = 3\pi \int_{-R}^{R} z^2 (R^2 - z^2) \, dz = \frac{4\pi R^5}{5}.$$

ΔΙΑΝΥΣΜΑΤΑ



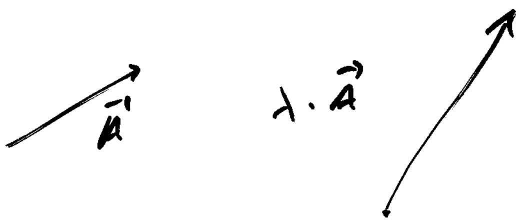
Πρόσθεση:





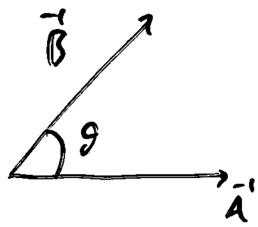
Πολ/σμοί:

με αριθμό:



Διανυσμάτων μεταξύ τους:

1) Εσωτερικό γινόμενο:



$$\vec{A} \cdot \vec{B} = |A| \cdot |B| \cdot \cos\theta$$

Ανάλυση διανύσματος σε βάση:

Βάση $\hat{x}, \hat{y}, \hat{z} \longrightarrow \hat{e}_i \quad i = 1, 2, 3$

$\downarrow \quad \downarrow \quad \downarrow$

$\hat{e}_1 \quad \hat{e}_2 \quad \hat{e}_3$

$$\vec{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z} = A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3$$

$$A_1 = A_x \quad A_2 = A_y, \quad A_3 = A_z$$

βάση ορθοκανονική:

$$\hat{e}_i \cdot \hat{e}_j = \delta_{ij} \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

$$\left.\begin{aligned} \vec{A} &= \sum_i A_i \hat{e}_i \\ \underline{A_i} &\underline{= \vec{A} \cdot \hat{e}_i} \end{aligned}\right\} \Rightarrow \quad \underline{\vec{A} = \sum_i (\vec{A} \cdot \hat{e}_i)\, \hat{e}_i}$$

Σύμβαση Einstein: επαναλαμβανόμενοι δείκτες αθροίζονται:

$$\vec{A} = \underline{\underline{A_i \hat{e}_i}} \equiv \sum_{i=1}^{3} A_i \hat{e}_i = A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3$$

Εσωτερικό γινόμενο:

$$\boxed{\vec{A} \cdot \vec{B} \equiv A_i B_i} = A_1 B_1 + A_2 B_2 + A_3 B_3$$

$|A|$: μέτρο του διανύσματος:

$$\vec{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z}$$

$$|A| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

Εξωτερικό γινόμενο:

$\underline{\vec{A} \times \vec{B}}$. διάνυσμα (κανόνας δεξιού χεριού)

x:

$$(\vec{A} \times \vec{B})_i = \epsilon_{ijk} A_j B_k$$

π.χ.: $$(\vec{A} \times \vec{B})_x = \epsilon_{xyz} A_y B_z + \epsilon_{xzy} A_z B_y$$

$$= A_y B_z - A_z B_y$$

Ιδιότητα:

$$\epsilon_{ijk} \epsilon_{ilm} = \delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}$$

$$\epsilon_{ijk} \epsilon_{ijl} = 2 \delta_{kl}$$

$$\epsilon_{ijk} \epsilon_{ijk} = 3!$$

Ισχύει:

$$\bar{a}\cdot(\bar{b}\times\bar{c}) = \bar{b}\cdot(\bar{c}\times\bar{a}) = \bar{c}\cdot(\bar{a}\times\bar{b})$$

$$\vec{a}\times(\bar{b}\times\bar{c}) = (\bar{a}\cdot\bar{c})\bar{b} - (\bar{a}\cdot\bar{b})\bar{c}$$

$$(\bar{a}\times\bar{b})\cdot(\bar{c}\times\bar{d}) = (\bar{a}\cdot\bar{c})(\bar{b}\cdot\bar{d}) - (\bar{a}\cdot\bar{d})(\bar{b}\cdot\bar{c})$$

ΑΣΚΗΣΗ: Να αποδειχτούν οι παραπάνω σχέσεις.

Επίσης:

$$\vec{A}\times(\vec{B}\times\vec{C}) + \vec{C}\times(\vec{A}\times\vec{B}) + \vec{B}\times(\vec{C}\times\vec{A}) = 0$$

$$(\vec{A}\times\vec{B})\times(\vec{C}\times\vec{D}) = \bar{C}(\bar{A}\times\bar{B}\cdot D) - \vec{D}(\bar{A}\times\bar{B}\cdot\bar{C})$$

$$g_{\rho\sigma} = g_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma$$

$$-1 = \Lambda^\mu{}_0 \Lambda^\nu{}_0 g_{\mu\nu}$$

$$= -(\Lambda^0{}_0)^2 + (\Lambda^i{}_0)^2 \cancel{\geq 1}$$

$$(\Lambda^0{}_0)^2 \geq 1$$

$$\Lambda^0{}_0 \geq 1$$

$$[M_{ij}, M_{kl}] = i\delta_{ik} M_{jl} - \delta_{il} M_{jk} - i\delta_{jk} M_{il} + \delta_{jl} M_{ik}$$

$$J_i = \tfrac{1}{2} \epsilon_{ijk} M_{jk}$$

$$K_i = M_{0i}$$

$$[J_i, J_j] = i \epsilon_{ijk} J_k$$

$$[J_i, K_j] = i \epsilon_{ijk} K_k$$

$$[K_i, K_j] =$$

$$N_i = \tfrac{1}{2}(J_i + i K_i)$$

$$N_i^\dagger$$

$$[N_i, N_j^\dagger] = 0$$

$$[N_i, N_j] = i \epsilon_{ijk} N_k \qquad su(2) \times su(2)$$

$(\sim, 0)$

$($

(j_0, j_1)

$(j_1, \ldots$

$(0, 0)$ 1st

$(1/4, 0)$ 2nd

$(0, 1/2)$ 4th

$(1/2, 1/2)$ see Ap