4. A conical surface (an empty ice-cream cone) carries a uniform charge density σ. The height of the cone is a, as is the radius of the top. Find the potential at point P (in the center of the opening of the cone), letting the potential at infinity be zero.

Solution:

The cone that I'll be considering has its vertex at the origin and opens up on the positive z-axis. I'll use cylindrical coordinates in my solution. Starting with the general expression for calculating potential:

$$V(\vec{r}) = \int\int \frac{k\,\sigma\,da'}{|\vec{r} - \vec{r}\,'|}$$

then I'll use what I know. For a cone with height h and base radius s, the equation of the cone is $z = \frac{h}{s}r$ so that

$$d\vec{r}_1 = r\,d\phi\,\hat{\phi} \qquad d\vec{r}_2 = dr\,\hat{r} + dz\,\hat{z} = (\frac{s}{h}\,\hat{r} + \hat{z})\,dz$$

which in turn yields

$$da = |d\vec{r}_1 \times d\vec{r}_2| = \frac{s}{h^2}\sqrt{s^2 + h^2}\,z\,dz\,d\phi$$

In this problem, the height and the radius of the base are the same value a, so the differential area element is:

$$da' = \sqrt{2}\,z'\,dz'\,d\phi'$$

Using this differential area element, and inserting the expression discussed in class (and in previous HW) for the distance $|\vec{r} - \vec{r}\,'|$ in cylindrical coordinates, the potential becomes:

$$V(\vec{r}) = \int_0^{2\pi}\int_0^{a} \frac{\sqrt{2}\,k\,\sigma\,z'\,dz'\,d\phi'}{\sqrt{r^2 + r'^2 - 2rr'\cos(\phi - \phi') + (z - z')^2}}$$